RECONSTRUCTION OF THE SINGULARITIES OF A POTENTIAL FROM BACKSCATTERING DATA IN 2D AND 3D

JUAN MANUEL REYES[1] AND ALBERTO RUIZ[2]

[1]Departamento de Matemáticas y Computación
Universidad de La Rioja
26004 Logroño, Spain
E-mail: juanmanuel.reyes@unirioja.es

[2]Departamento de Matemáticas
Universidad Autónoma de Madrid
28049 Madrid, Spain
E-mail: alberto.ruiz@uam.es

Abstract. We prove that the singularities of a potential in the two and three dimensional Schrödinger equation are the same as the singularities of the Born approximation (Diffraction Tomography), obtained from backscattering inverse data, with an accuracy of $1/2^-$ derivative in the scale of L^2-based Sobolev spaces. This improves previous results, see [28] and [18], removing several constrains on the a priori regularity of the potential. The improvement is based on the study of the smoothing properties of the quartic term in the Neumann-Born expansion of the scattering amplitude in 3D, together with a Leibniz formula for multiple scattering valid in any dimension.

1. Introduction

The inverse scattering problem for Schrödinger potentials deals with the uniqueness, reconstruction and stability of the potential q in the Hamiltonian $H = \Delta + q$ from the far field pattern of the generalized eigenfunctions or scattering solutions. These are the unique solutions of the asymptotic boundary value problem

$$\begin{cases} (\Delta + q + k^2)u = 0 \\ u = e^{ikx\cdot\theta} + u_{out}, \end{cases} \tag{1.1}$$

where the function u_{out} satisfies the outgoing Sommerfeld radiation condition, which means, for compactly supported potential q, that u has asymptotics as $|x| \to \infty$

$$u(x,\theta,k) = e^{ikx.\theta} + C|x|^{\frac{1-n}{2}} k^{\frac{n-3}{2}} e^{ik|x|} A(\theta',\theta,k) + o(|x|^{\frac{1-n}{2}}), \tag{1.2}$$

where $\theta' = x/|x|$.

Both authors were supported by Spanish Grant MTM2005-07652-C02-01.

The function $A(\theta', \theta, k)$, $k > 0$, θ, θ' in the unit sphere S^{n-1}, is known as scattering amplitude or far field pattern.

The inverse problem for whole data is formally overdeterminate, as one easily can see by counting variables. For this reason, to avoid redundancies, some kinds of partial data are selected for the inverse problems. The selection of these data is motivated by numerical experience and applications. The most celebrated sets of partial data are the following:

- Fixed energy data. We assume as data the values $A(\theta', \theta, k)$ for fixed k and free $\theta, \theta' \in \mathbb{S}^{n-1}$. Uniqueness of the inverse problem in this case was studied by [16], [17], [23], [32]. The approach to this problem is related to the Calderón- Sylvester-Uhlmann complex exponential solutions, used in the Electrical Impedance Tomography inverse problem. The stability happens to be very weak.
- Fixed angle data. The knowledge of $A(\theta', \theta, k)$ for fixed θ, free $\theta' \in \mathbb{S}^{n-1}$ and free $k > 0$ is assumed. The uniqueness of the inverse problem is open and only generic and local uniqueness is proved under a priori regularity assumptions on the potential, see [30].
- Backscattering data. One assumes $A(-\theta, \theta, k)$ for free $\theta \in \mathbb{S}^{n-1}$ and free $k > 0$. The uniqueness of the inverse problem is only proved generically and for small potentials, see [7], see also [22], [30], [14].

In practical applications the actual potential is substituted by the so called Born approximation of the scattering amplitude. The procedures to imaging the Born approximation from the scattering data are known as Diffraction Tomography.

The different Born approximations are obtained, in the frequency domain, from the formula

$$\hat{q}_{approx}(\xi) = A(\omega, \theta, k),$$

where ξ is given by the redundant relation

$$\xi = k(\omega - \theta).$$

If θ is fixed (fixed angle data), we use the change of variable $\xi = \Phi_\theta(k, \omega) = k(\omega - \theta)$ to define the Born approximation $\hat{q}_\theta(\xi) = A(\omega, \theta, k)$. Notice that this change of variable becomes singular on the hyperplane $\xi \cdot \theta = 0$.

For backscattering data the Born approximation is given in the frequency domain by the polar coordinates

$$\hat{q}_B(-2k\theta) = A(-\theta, \theta, k). \tag{1.3}$$

This fact makes backscattering data more natural and simpler than fixed angle data for diffraction tomography.

The use of the Born approximation is not, in general, justified on a mathematical basis: one would like to know how much information on the actual potential q is contained in the Born approximation. This problem has been treated by several authors. In full data case see [19], [21], [20] and [2]. For fixed angle data and backscattering data, both of which are formally well determinate, the justification of diffraction tomography was studied in [26] (fixed angle) and [11], [18], [28], [24]

(backscattering). We would like to remark that each of the last two types of data require the analysis of special multilinear operators which are not related.

In this work we study the case of backscattering data in dimension two and three, we continue and complete the research of [18], [28] and [24], by removing some constrains in their results. We prove that the diffraction tomography is a migration scheme, see [3], within an accuracy of at least 1/2. This is to say that the most singular parts of the actual potential can be reconstructed from the Born approximation up to a certain order (the accuracy of the migration). The determination of this accuracy is very important to design numerical methods, adapted to the spaces in which one expects to obtain the information on the actual potential from real scattering data. We prove

Theorem 1. *Assume that $n \in \{2,3\}$, q is a compactly supported function in $W^{\alpha\,,2}(\mathbb{R}^n)$ and $\alpha \geq 0$. Then $q - q_B \in W^{\beta\,,2}(\mathbb{R}^n) + C^\infty(\mathbb{R}^n)$, for any $\beta \in \mathbb{R}$ such that $0 \leq \beta < \alpha + \frac{1}{2}$.*

In Theorem 1 the regularity is measured in the scale of L^2-based Sobolev spaces. The optimality of this accuracy in this scale of spaces is, so far, an open and interesting question.

The procedure to justify the migration scheme is to study the smoothing properties of the multilinear terms in the Neumann-Born expansion of the scattering amplitude (multiple scattering).

Physical evidence suggests that multiple scattering is strong in the case of backscattering data. The control of double and triple scattering in 3D, within an accuracy of 1/2, was obtained in [28], but their result together with the general estimates for multiple scattering do not suffice to assure that, for a potential q a priori in the Sobolev space $W^{\alpha,2}$, the error $q - q_B$ is in $W^{\beta,2}$ for any $\beta < \alpha + 1/2$; the restriction $0 \leq \alpha < 3/4$ is needed. In the range $\alpha \geq 3/4$ known estimates of quadruple scattering became worse than those of double or triple scattering. The study of the accuracy of the Born approximation requires, then, to improve the estimates of the quartic term in the series. We accomplish this in the present work. We also extend the results, which previously were only studied for $\alpha < 3/2$ in 3D and for $\alpha < 1$ in 2D, to any $\alpha \geq 0$ by using a Leibniz' type formula for the derivatives of multiple scattering terms (see §C.1 in [25]).

In dimension three, we only are able to prove that the errors due to double, triple and quadruple scattering are a half of a derivative better than the actual potential, as opposite to the 2D case where the regularity increases with the order.

Result from [28], [24] together with Corollary 1 allow us to state the following result concerning reconstruction of classical discontinuities from backscattering in 2D:

Theorem 2. *Let q compactly supported in $W^{\alpha,2}(\mathbb{R}^2)$, where $\alpha > 0$. Then $q - q_B$ is a continuous function.*

In fact, it was proved (Theorem 2 in [28]), that, for such a q, the quadratic term is a continuous function. Hölder continuity of the cubic term is obtained since it is in $W^{\beta,2}$ for all $\beta < \alpha + 1$, see Theorem 1 in [24]. The remainder is controled by Corollary 1.

In the three dimensional case, it follows from Theorem 1 that the whole non continuous part of the actual potential can be reconstructed from the Born approximation, assuming a priori that q is

in the Sobolev space $W^{\alpha,2}$ for some $\alpha > 1$. Notice q might have some discontinuities if α is between 1 and the 3D Sobolev exponent 3/2:

Theorem 3. *Let q compactly supported in $W^{\alpha,2}(\mathbb{R}^3)$, where $\alpha > 1$. Then $q - q_B$ is a continuous function.*

From the previous work [28] it follows that in 3D the discontinuities in the case of a piecewise regular potential can be reconstructed from the Born approximation (the result is not stated in [28] but it is similar to Corollary 0.1 in [18] in the 2D case). By using the evolution equation the reconstruction of conormal singularities was achived in [11]. On one hand Theorem 3, as far as we know, is the first result of reconstruction of discontinuities in 3D, without assuming special structure of the singular set but, on the other hand, one expects that $q \in W^{\alpha,2}$, for any $\alpha > 1/2$ suffices for the reconstruction of discontinuities. So far, this improvement has not been achieved. We know from Corollary 1, that the high frequency Neumann-Born series for $j \geq 5$ converges to a Hölder continuous function for $\alpha > \frac{1}{2}$.

An important feature of Theorem 1 is the fact that, regardless of the a priori regularity assumptions on the potential, the accuracy of the migration scheme is at least 1/2. This independency is important to construct any recurrence scheme, in order to obtain further information on the actual potential from scattering data. In the case of fixed angle data, one can define a modified Born approximation by inserting the error $q - q_B$ in the quadratic form, see [26]. This increases the known accuracy for rough potentials q, but an inconvenient to iterate the procedure is the dependency on α of the accuracy.

Finally, we remark that in the higher dimensional case, the order of accuracy is an open question. We believe that 1/2 also applies, but the technical complexity of our approach makes it necessary to look for a new point of view on the problem. The treatment of the 3D problem due to Lagergren [14], [15], based upon a time dependent expansion of the backscattering operator, also requires a very technical treatment of its multilinear term. See also [4] and [5].

Notation and definitions. We will write $\mathcal{F}f$ or $\hat{f}$ to denote the Fourier transform of f. $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. The letter M denotes the Hardy-Littlewood maximal operator. We denote the two-dimensional Hausdorff measure in $\mathbb{R}^3$ by σ. The expression $|x| \sim 2^{-j}|\eta|$ refers to $2^{-j-1}|\eta| < |x| \leq 2^{-j+1}|\eta|$, for $j \in \mathbb{Z}$, $x, \eta \in \mathbb{R}^3$. We will use the homogeneous and non homogeneous Hilbertian Sobolev spaces. With $\alpha \in \mathbb{R}$, we denote

$$\dot{W}^{\alpha,2}(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : |\cdot|^\alpha \mathcal{F}f(\cdot) \in L^2(\mathbb{R}^n)\},$$
$$W^{\alpha,2}(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : (1+|\cdot|^2)^{\alpha/2} \mathcal{F}f(\cdot) \in L^2(\mathbb{R}^n)\}.$$

Let $\eta,\, \xi \in \mathbb{R}^3 \setminus \{0\}$. We write

$$\Gamma(\eta) := \left\{ x \in \mathbb{R}^3 : \left| x - \frac{\eta}{2} \right| = \frac{|\eta|}{2} \right\}, \tag{1.4}$$

refering to the sphere centered at $\frac{\eta}{2}$ and radius $\frac{|\eta|}{2}$ and

$$\Lambda(\xi) := \left\{ x \in \mathbb{R}^3 : \xi \cdot (x - \xi) = 0 \right\} \tag{1.5}$$

denotes the plane orthogonal to ξ that contains the point ξ. We denote $\Gamma(\eta)^3 := \Gamma(\eta) \times \Gamma(\eta) \times \Gamma(\eta)$ and $\Gamma(\eta)^2 := \Gamma(\eta) \times \Gamma(\eta)$.

Let $\widehat{F}(\eta)$ given by the integral on a manifold $A(\eta)$ of some function. Since our proofs are based upon a decomposition of $A(\eta)$ in several subdomains $D(\eta) \subset A(\eta)$, we will denote by $\widehat{F}_D(\eta)$ the same expression when we restrict the integration to the subdomain $D(\eta)$.

The outgoing resolvent operator for the Laplacian is defined, in terms of the Fourier transform, by

$$\widehat{R_k(f)}(\xi) = (-|\xi|^2 + k^2 + i0)^{-1} \widehat{f}(\xi) \,.$$

We define the operator Q_j in the following way

$$\widehat{Q_j(q)}(\xi) = \int_{\mathbb{R}^n} e^{ik\theta \cdot y} (qR_k)^{j-1} (q(\cdot) e^{ik\theta \cdot (\cdot)})(y) dy \,, \tag{1.6}$$

where $k = |\xi|/2$, $\theta = -\xi/|\xi|$. With these expressions for k and θ, we define the multi-linear form $Q_j(f_1, \dots, f_j)$ in the FT side as

$$\mathcal{F}\left(Q_j(f_1, \dots, f_j)\right)(\xi) := \int_{\mathbb{R}^n} e^{ik\theta \cdot y} (f_1 R_k)(f_2 R_k) \dots (f_{j-1} R_k)(f_j(\cdot) e^{ik\theta\cdot(\cdot)})(y) dy.$$

We denote the high frequency version

$$\widetilde{Q}_j(q) := \mathcal{F}^{-1}(\chi^*(|\cdot|/2) \widehat{Q_j(q)}(\cdot)) \,, \tag{1.7}$$

where $Q_j(q)$ is defined in (1.6) and $\chi^* \in C^\infty(\mathbb{R})$ with $\chi^*(t) = 1$ if $t \geq 2C_0$, $\chi^*(t) = 0$ if $t < C_0$, for a certain constant $C_0 > 0$ to be chosen (see section 4). Notice that the cutoff near the origin allows us to reduce the estimates of Sobolev norms to the estimation of homogeneous Sobolev norms.

We also write $\widetilde{Q}_j(f_1, \dots, f_j) = \mathcal{F}^{-1}\left(\chi^* \mathcal{F}\left(Q_j(f_1, \dots, f_j)\right)\right)$. We will admit the abuse of notation $Q_j(q) = Q_j(q, \dots, q)$ and $\widetilde{Q}_j(q) = \widetilde{Q}_j(q, \dots, q)$.

The permutation group of order k is denoted by S_k. For multi-indexes β and γ in $\mathbb{N}^n$, we use the standard definitions of $\beta!$, $|\beta|$ and $\beta \leq \gamma$.

We use the letter C to denote any constant that can be explicitly computed in terms of known quantities. The exact value denoted by C may change from line to line in a given computation.

2. Preliminaries and results

We obtain the so called Lippmann-Schwinger integral equation by applying the outgoing resolvent to (1.1)

$$u(x, \theta, k) = e^{ikx\cdot\theta} + R_k(q(\cdot) u(\cdot, \theta, k))(x) \,. \tag{2.1}$$

The key operator in the above integral equation is

$$T_k(f)(x) = R_k(q(\cdot) f(\cdot))(x) \,.$$

There are several a priori estimates for R_k that allow to prove existence and uniqueness of Lippmann-Schwinger integral equation. Usually, Fredholm theory applies and everything follows from compactness arguments, Rellich uniqueness theorem and unique continuation principles, in the case of real valued potentials. The solution can be obtained in several situations (these cases do not require q to be real) by perturbation arguments, assuming that the energy is sufficiently large, $k > k_0 \geq 0$, where

k_0 depends on some a priori bound of the potential q. As an example we may consider compactly supported $q \in L^r(\mathbb{R}^n)$ for some $r > \frac{n}{2}$. In this case, which is the one considered in this work, the resolvent operator R_k is bounded from $L^p(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$ with norm decaying to 0 as $k \to \infty$ when $\frac{1}{p} - \frac{1}{p'} = \frac{1}{r}$, see [1], [13] and see also [26]. This together with Hölder inequality proves that for big k the operator T_k is a contraction in L^p and then existence and uniqueness of solution of (2.1) easily follow and u can be expressed as a convergent Neumann-Born series.

Once the scattering solution is obtained we may prove that the far field pattern can be expressed as

$$A(\theta', \theta, k) = \int_{\mathbb{R}^n} e^{-ik\theta' \cdot y} q(y) u(y, \theta, k) dy \,, \tag{2.2}$$

see [7] where this is used as a definition for non compactly supported potentials.

By inserting the series u in (2.2) one obtains the Neumann-Born series of the scattering amplitude for k large enough (high frequency Born series):

$$A(\theta', \theta, k)\chi^*(k) = \hat{q}(k(\theta' - \theta))\chi^*(k) + \sum_{j=2}^{\infty} \mathrm{q}_j(q)(\theta', \theta, k) \,, \tag{2.3}$$

where

$$\mathrm{q}_j(q)(\theta', \theta, k) = \chi^*(k) \int_{\mathbb{R}^n} e^{-ik\theta' \cdot y} (qR_k)^{j-1}(q(\cdot)e^{ik\theta\cdot(\cdot)})(y) dy \,,$$

and χ^* is a cutoff function near the origen (see the notations).

We deal with the backscattering inverse problem, for which one assumes the data with the direction of the receiver opposed to the source direction (echoes), i.e. $A(-\theta, \theta, k)$. The inverse problem is then formally well determined. In this case the Neumann-Born series for the scattering amplitude is

$$A(-\theta, \theta, k)\chi^*(k) = \hat{q}(\xi)\chi^*(k) + \sum_{j=2}^{\infty} \widehat{\widetilde{Q}_j(q)}(\xi) \,, \tag{2.4}$$

where $\xi = -2k\theta$ and the j-adic term in the Neumann-Born series $\widetilde{Q}_j(q)$ is given by the operator (1.7). We define the Born approximation for high frequency backscattering data as

$$\widehat{q_{B,H}}(\xi) = A(-\theta, \theta, k)\chi^*(k)$$

where $\xi = -2k\theta$. Notice that the series (2.4) is addapted to the reconstruction of singularities, since $q_{B,H} - q_B$ and $q - \mathcal{F}^{-1}(\hat{q}(\cdot)\chi^*(|\cdot|/2))$ are C^∞ functions.

We denote the remainder term in the high frequency series as

$$\mathbf{R}_l(q) = \sum_{j=l}^{\infty} \widetilde{Q}_j(q).$$

The main part of this work, which is §3, is due to obtain the control of the term $\widetilde{Q}_4(q)$ in dimension three:

Theorem 4. *Let us assume that q is a compactly supported function in $W^{\alpha,2}(\mathbb{R}^3)$, for $0 \le \alpha < 3/2$. Then $\widetilde{Q}_4(q) \in W^{\beta,2}(\mathbb{R}^3)$, for any β such that $0 \le \beta < \alpha + 1/2$.*

We also prove in section §4:

Theorem 5. *Let $n \in \{2,3\}$, $q \in W^{\alpha,2}(\mathbb{R}^n)$ compactly supported and $0 \leq \alpha < n/2$. Assume that $C_0 > \max\{(2\|q\|_{W^{\alpha,2}})^4, 1\}$, $l = 4$ if $n = 2$ and $l = 5$ if $n = 3$, see (1.7). Then, for any $\beta \in \mathbb{R}$ such that $\beta < \alpha + 1$ the remainder in the high frequency Born series $\mathbf{R}_l$ converges to a function in $W^{\beta,2}(\mathbb{R}^n)$.*

From Sobolev embedding theorem we obtain,

Corollary 1. In the hypothesis of Theorem 5, assume also $\alpha > 0$ in 2D and $\alpha > \frac{1}{2}$ in 3D. Then $\mathbf{R}_l$ is a Hölder continuous function.

Theorem 1 in the case $0 \leq \alpha < n/2$ will follow from the above theorems, together with the following estimates for the quadratic and cubic terms, see [28] and [24]:

$$\|\widetilde{Q}_2(q)\|_{\dot{W}^{\beta,2}} \leq C\, \|q\|_{\dot{W}^{-\frac{1}{2},2}} \|q\|_{\dot{W}^{\alpha,2}}\,, \tag{2.5}$$

$$\|\widetilde{Q}_3(q)\|_{\dot{W}^{\beta,2}} \leq C \left(\|q\|_{L^2}^2 + \|q\|_{L^2} \|q\|_{\dot{W}^{-\frac{1}{2}-\varepsilon,2}} + \|q\|_{\dot{W}^{-\frac{1}{2},2}} \|q\|_{\dot{W}^{-\varepsilon,2}} \right) \|q\|_{\dot{W}^{\alpha,2}}\,, \tag{2.6}$$

where the dimension is $n = 3$, $\beta < \alpha + 1/2$, $\varepsilon := \alpha + \frac{1}{2} - \beta > 0$. For dimension $n = 2$ we have (2.5) for $\beta < \alpha + 1/2$ and

$$\|\widetilde{Q}_3(q)\|_{\dot{W}^{\beta,2}} \leq C \left(\|q\|_{L^2} \|q\|_{\dot{W}^{-\frac{1}{2},2}} + \|q\|_{L^2}^2 \right) \|q\|_{\dot{W}^{\alpha\,,\,2}}\,, \tag{2.7}$$

when $0 \leq \beta < \alpha + 1$.

In §5 we give the procedure to extend the above results to the case $\alpha \geq n/2$. The key is Theorem 6 which is a Leibniz' type formula for derivatives of multiple scattering terms.

The proofs are very involved and technical. For this reason, we only include the details of the proof in the key case of Theorem 4, see Proposition 3 in §3.1. In other cases, we just sketch the proof and try to convince the reader that similar arguments work.

3. Proof of Theorem 4.

Theorem 4 follows from the estimate

Proposition 1. *For q, α under the same hypothesis of Theorem 4 it holds*

$$\|\widetilde{Q}_4(q)\|_{\dot{W}^{\beta,2}} \leq C \Big(\|q\|_{L^2}^3 + \|q\|_{\dot{W}^{-\frac{1}{2},2}} \|q\|_{L^2}^2 + \|q\|_{\dot{W}^{-\frac{1}{2},2}} \|q\|_{\dot{W}^{-\varepsilon,2}} \|q\|_{L^2} \tag{3.1}$$

$$+ \|q\|_{L^2}^2 \|q\|_{\dot{W}^{-\frac{1}{2}-\varepsilon\,,\,2}} \Big) \|q\|_{\dot{W}^{\alpha,2}}, \tag{3.2}$$

for all $\beta \in \mathbb{R}$ such that $0 \leq \beta < \alpha + 1/2$, where $\varepsilon := \alpha + \frac{1}{2} - \beta > 0$.

The quartic term in the Neumann-Born series for backscattering data is given by

$$\widehat{Q_4(q)}(\xi) := \int_{\mathbb{R}^3} e^{ik\theta\cdot y} (qR_+(k^2))^3 (q(\cdot) e^{ik\theta\cdot(\cdot)})(y)\, dy\,,$$

for any $\xi \in \mathbb{R}^3$, where $\xi = -2k\theta$, that is, $k = \frac{|\xi|}{2}$ and $\theta = -\frac{\xi}{|\xi|}$. From Lemma 3.1 in [27], this term can written as

Proposition 2. *For any dimension n and $\eta \in \mathbb{R}^n \setminus \{0\}$,*

$$\widehat{Q_4(q)}(\eta) = p.v. \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\hat{q}(\xi)\hat{q}(\eta-\tau)\hat{q}(\tau-\phi)\hat{q}(\phi-\xi)}{[\xi\cdot(\eta-\xi)]\,[\tau\cdot(\eta-\tau)]\,[\phi\cdot(\eta-\phi)]}\, d\xi d\tau d\phi \tag{3.3}$$

$$+ 2\frac{i\pi}{|\eta|}\, p.v. \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\Gamma(\eta)} \frac{\hat{q}(\xi)\hat{q}(\eta-\tau)\hat{q}(\tau-\phi)\hat{q}(\phi-\xi)}{[\tau\cdot(\eta-\tau)]\,[\phi\cdot(\eta-\phi)]}\, d\sigma(\xi) d\tau d\phi \tag{3.4}$$

$$+ \frac{i\pi}{|\eta|}\, p.v. \int_{\Gamma(\eta)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\hat{q}(\xi)\hat{q}(\eta-\tau)\hat{q}(\tau-\phi)\hat{q}(\phi-\xi)}{[\xi\cdot(\eta-\xi)]\,[\tau\cdot(\eta-\tau)]}\, d\xi d\tau d\sigma(\phi) \tag{3.5}$$

$$- 2\frac{\pi^2}{|\eta|^2}\, p.v. \int_{\Gamma(\eta)}\int_{\Gamma(\eta)}\int_{\mathbb{R}^n} \frac{\hat{q}(\xi)\hat{q}(\eta-\tau)\hat{q}(\tau-\phi)\hat{q}(\phi-\xi)}{\xi\cdot(\eta-\xi)}\, d\xi d\sigma(\tau) d\sigma(\phi) \tag{3.6}$$

$$- \frac{\pi^2}{|\eta|^2}\, p.v. \int_{\mathbb{R}^n}\int_{\Gamma(\eta)}\int_{\Gamma(\eta)} \frac{\hat{q}(\xi)\hat{q}(\eta-\tau)\hat{q}(\tau-\phi)\hat{q}(\phi-\xi)}{\phi\cdot(\eta-\phi)}\, d\sigma(\xi) d\sigma(\tau) d\phi \tag{3.7}$$

$$- \frac{i\pi^3}{|\eta|^3} \int_{\Gamma(\eta)}\int_{\Gamma(\eta)}\int_{\Gamma(\eta)} \hat{q}(\xi)\hat{q}(\eta-\tau)\hat{q}(\tau-\phi)\hat{q}(\phi-\xi)\, d\sigma(\xi) d\sigma(\tau) d\sigma(\phi)\,. \tag{3.8}$$

The key to understand the structure of the quartic term is the pure spherical measures part (3.8). Hence we define, for any $\eta \in \mathbb{R}^3 \setminus \{0\}$

Notation:

$$\widehat{Q(q)}(\eta) := \frac{1}{|\eta|^3} \int_{\Gamma(\eta)}\int_{\Gamma(\eta)}\int_{\Gamma(\eta)} \hat{q}(\xi)\hat{q}(\eta-\tau)\hat{q}(\tau-\phi)\hat{q}(\phi-\xi)\, d\sigma(\xi) d\sigma(\tau) d\sigma(\phi)\,. \tag{3.9}$$

We prove in section §3.1:

Proposition 3. *Let $q \in W^{\alpha,2}(\mathbb{R}^3)$ be a compactly supported function with $0 \le \alpha < 3/2$. Then for all $\beta < \alpha + 1/2$,*

$$\|Q(q)\|_{\dot{W}^{\beta,2}} \le C \left(\|q\|_{L^2}^3 + \|q\|_{\dot{W}^{-\frac{1}{2},2}} \|q\|_{L^2}^2 + \|q\|_{\dot{W}^{-\frac{1}{2},2}} \|q\|_{\dot{W}^{-\varepsilon,2}} \|q\|_{L^2} \right.$$
$$\left. + \|q\|_{L^2}^2 \|q\|_{\dot{W}^{-\frac{1}{2}-\varepsilon\,,\,2}} \right) \|q\|_{\dot{W}^{\alpha,2}}\,,$$

where $\varepsilon := \alpha + \frac{1}{2} - \beta > 0$ and the constant $C > 0$ just depends of α, β, ϵ and the support of q.

Now we sketch the estimates of principal value terms (3.3)-(3.7) We use a decomposition of the space into diadic shelves, as it was done for the cubic term in 2D, see [24], and for the quadratic and cubic terms in 3D in [28]. More detail can be seen in [25].

Let us state, as a model, the main features to control the principal value term $Q'(q)$, given by (3.7),

$$\mathcal{F}(Q'(q))(\eta) := \frac{1}{|\eta|^2}\, p.v. \int_{\mathbb{R}^3}\int_{\Gamma(\eta)}\int_{\Gamma(\eta)} \frac{\hat{q}(\xi)\hat{q}(\eta-\tau)\hat{q}(\tau-\phi)\hat{q}(\phi-\xi)}{\phi\cdot(\eta-\phi)}\, d\sigma(\xi) d\sigma(\tau) d\phi. \tag{3.10}$$

The key to estimate this principal value operator is to control the term:

$$\mathcal{F}(Q_\delta(q))(\eta) := \chi_{(\delta^{-1},\infty)}(|\eta|) \frac{1}{|\eta|^4} \int_{\Gamma_\delta(\eta)}\int_{\Gamma(\eta)}\int_{\Gamma(\eta)} \hat{q}(\xi)\hat{q}(\eta-\tau)\hat{q}(\tau-\phi)\hat{q}(\phi-\xi)\, d\sigma(\xi) d\sigma(\tau) d\phi, \tag{3.11}$$

where

$$\Gamma_\delta(\eta) := \{\phi \in \mathbb{R}^3 : ||\phi - \eta/2| - |\eta|/2| \le \delta|\eta|\} \tag{3.12}$$

.

Comparing (3.11) with (3.9), we observe that we replace the sphere $\Gamma(\eta)$, in which the variable ϕ runs, by its tubular neighborhood $\Gamma_\delta(\eta)$ of width $\delta|\eta|$ in the normal direction. Notice that $d\sigma_\eta(\phi) = \lim_{\delta\to 0} \frac{1}{\delta|\eta|}\, \chi_{\Gamma_\delta(\eta)}(\phi)\, d\phi$, where $d\sigma_\eta(\phi)$ denotes the measure on the sphere $\Gamma(\eta)$ induced by Lebesgue measure $d\phi$. For δ small, we have that $\widehat{Q_\delta(q)}(\eta) \sim \chi_{(\delta^{-1},\infty)}(|\eta|)\, \delta\, \widehat{Q(q)}(\eta)$. In this way we may expect estimates for the Sobolev norm of $Q_\delta(q)$ obtained from estimates of $Q(q)$ multiplied by δ. If one follows the lines of the proof of Proposition 3, one gets the following

Lemma 3.1. *Assume that $q \in W^{\alpha,2}(\mathbb{R}^3)$ is compactly supported and $0 \le \alpha < 3/2$. Let $\beta < \alpha + 1/2$ and $\varepsilon = \alpha + 1/2 - \beta > 0$. Then there exist $\delta_1 > 0$ and $\gamma = \gamma(\epsilon) > 1$ such that for all δ satisfying $0 < \delta \le \delta_1$ it holds*

$$\|Q_\delta(q)\|_{\dot W^{\beta,2}} \le C\delta^\gamma \Big(\|q\|_{L^2}^3 + \|q\|_{\dot W^{-\frac12,2}}\|q\|_{L^2}^2 + \|q\|_{\dot W^{-\frac12,2}}\|q\|_{\dot W^{-\varepsilon,2}}\|q\|_{L^2} \tag{3.13}$$

$$+ \|q\|_{L^2}^2\|q\|_{\dot W^{-\frac12-\varepsilon\,,\,2}}\Big) \|q\|_{\dot W^{\alpha,2}}, \tag{3.14}$$

where C just depends on α, β, δ_1 and the support of q.

Now, to estimate the term (3.10), we use a decomposition of the Euclidean space $\mathbb{R}^3$ in a similar way as was done in [24] for 2D:

$$\mathbb{R}^3 = \Gamma_{j_1^-}(\eta) \cup \bigcup_{j=j_1}^{[\log_2|\eta|]} \Gamma_j(\eta) \cup \Gamma_\infty^*(\eta), \tag{3.15}$$

where j_1 is the lowest integer such that $j_1 \ge 1 - \log_2(\delta_1)$ with δ_1 from Lemma 3.1, $|\eta| \ge 2^{j_1-1}$ and

$$\begin{aligned}
&\Gamma_{j_1^-}(\eta) := \{\phi \in \mathbb{R}^3 : ||\phi - \eta/2| - |\eta|/2| > 2^{-j_1+1}|\eta|\},\\
&\Gamma_j(\eta) := \{\phi \in \mathbb{R}^3 : ||\phi - \eta/2| - |\eta|/2| \sim 2^{-j}|\eta|\}, \quad j_1 \le j \le [\log_2|\eta|],\\
&\Gamma_\infty^*(\eta) := \{\phi \in \mathbb{R}^3 : ||\phi - \eta/2| - |\eta|/2| \le 2^{-[\log_2|\eta|]-1}|\eta|\}.
\end{aligned}$$

Remark. Technically this partition only makes sense for $j_1 \ge 3$, but this is not a constraint if we demand $\delta_1 \le 1/4$, since $j_1 \ge 1 - \log_2(\delta_1)$. Notice that $\Gamma_\infty^*(\eta) \subset \Gamma_\infty(\eta)$, where

$$\Gamma_\infty(\eta) := \{\phi \in \mathbb{R}^3 : ||\phi - \eta/2| - |\eta|/2| < 1\}.$$

This decomposition is used to split the operator (3.10). To control the operator corresponding to the annulus terms Lemma 3.1, with $\delta = 2^{-j+1}$, suffices. To deal with the central term, corresponding to $\Gamma_\infty^*(\eta)$, which is close to the singularity $\Gamma(\eta)$, we use again Lemma 3.1 and the following

Lemma 3.2. *Let $f_j \in W^{\alpha,2}(\mathbb{R}^3)$, $j = 1, \dots, 4$, be functions such that f_1, f_2 are compactly supported and $0 \le \alpha < 3/2$. We denote*

$$\mathcal{F}(Q_\infty^*(f_1,f_2,f_3,f_4))(\eta) := \chi^*(\eta)\frac{1}{|\eta|^3}\int_{\Gamma_\infty(\eta)}\int_{\Gamma(\eta)}\int_{\Gamma(\eta)} |\widehat{f_1}(\xi)\widehat{f_2}(\eta-\tau)| \tag{3.16}$$

$$\times |\widehat{f_3}(\tau-\phi)\widehat{f_4}(\phi-\xi)|\, d\sigma(\xi)d\sigma(\tau)d\phi. \tag{3.17}$$

Then for any $\beta < \alpha + 1/2$,

$$\|Q_\infty^*(f_1,f_2,f_3,f_4)\|_{\dot W^{\beta,2}}$$

$$\le C(\alpha,\beta,\operatorname{supp} f_1,\operatorname{supp} f_2)\,\Big(\sum_{\sigma\in S_4} \|f_{\sigma(1)}\|_{L^2}\|f_{\sigma(2)}\|_{L^2}\|f_{\sigma(3)}\|_{L^2}\|f_{\sigma(4)}\|_{\dot W^{\alpha,2}} \tag{3.18}$$

$$+ \sum_{\tau\in S_4} \|f_{\tau(1)}\|_{\dot W^{-\frac12,2}} \|f_{\tau(2)}\|_{L^2} \|f_{\tau(3)}\|_{L^2} \|f_{\tau(4)}\|_{\dot W^{\alpha,2}} \tag{3.19}$$

$$+ \sum_{\omega\in S_4} \|f_{\omega(1)}\|_{\dot W^{-\frac12,2}} \|f_{\omega(2)}\|_{\dot W^{-\varepsilon,2}} \|f_{\omega(3)}\|_{L^2} \|f_{\omega(4)}\|_{\dot W^{\alpha,2}} \tag{3.20}$$

$$+ \sum_{\rho\in S_4} \|f_{\rho(1)}\|_{\dot W^{-\frac12-\varepsilon,2}} \|f_{\rho(2)}\|_{L^2} \|f_{\rho(3)}\|_{L^2} \|f_{\rho(4)}\|_{\dot W^{\alpha,2}} \,), \tag{3.21}$$

where $\varepsilon := \alpha + \frac12 - \beta > 0$.

To estimate this central term, dealing with the principal value, one needs to use the cancelation. We must replace the integral on the ring $\Gamma^*_\infty(\eta)$ by $\int_{\Gamma^+_\varepsilon(\eta)} + \int_{\Gamma^-_\varepsilon(\eta)}$, where

$$\Gamma^+_\varepsilon(\eta) = \{\xi \in \mathbb{R}^3 : \varepsilon < |\xi - \eta/2| - |\eta|/2 < 2^{-[\log_2|\eta|]-1}|\eta|\}$$
$$\Gamma^-_\varepsilon(\eta) = \{\xi \in \mathbb{R}^3 : \varepsilon < |\eta|/2 - |\xi - \eta/2| < 2^{-[\log_2|\eta|]-1}|\eta|\}.$$

The map $F : \Gamma^+_\varepsilon(\eta) \to \Gamma^-_\varepsilon(\eta)$, given by symmetry with respect to $\Gamma(\eta)$ allows us to pass to the limit when $\varepsilon \to 0^+$. To cancel the singularities we use an estimate, due to Calderón, for first differences in terms of the Hardy-Littlewood maximal operator M (as in [24], several standard reductions are also needed):

Lemma 3.3 (see [12]). *Let* $u \in W^{1,p}(\mathbb{R}^n)$, $p > 1$, $a \in \mathbb{R}^n$. *Then*

$$|u(x) - u(x-a)| \le C\,|a|\,[M(\nabla u)(x) + M(\nabla u)(x-a)]. \tag{3.22}$$

After some changes of variables in the integrals involving F, we reduce to study the following terms:

$$\frac{1}{|\eta|^3}\int_{\Gamma_\infty(\eta)}\int_{\Gamma(\eta)}\int_{\Gamma(\eta)} |\hat q(\xi)\hat q(\eta-\tau)\hat q(\tau-\phi)| M\nabla\hat q(\phi-\xi)\, d\sigma(\xi) d\sigma(\tau) d\phi,$$
$$\frac{1}{|\eta|^3}\int_{\Gamma_\infty(\eta)}\int_{\Gamma(\eta)}\int_{\Gamma(\eta)} |\hat q(\xi)\hat q(\eta-\tau)| M\hat q(\tau-\phi) M\nabla\hat q(\phi-\xi)\, d\sigma(\xi) d\sigma(\tau) d\phi,$$
$$\frac{1}{|\eta|^3}\int_{\Gamma_\infty(\eta)}\int_{\Gamma(\eta)}\int_{\Gamma(\eta)} |\hat q(\xi)\hat q(\eta-\tau)| M\nabla\hat q(\tau-\phi) M\hat q(\phi-\xi)\, d\sigma(\xi) d\sigma(\tau) d\phi,$$
$$\frac{1}{|\eta|^3}\int_{\Gamma_\infty(\eta)}\int_{\Gamma(\eta)}\int_{\Gamma(\eta)} |\hat q(\xi)\hat q(\eta-\tau)| M\nabla\hat q(\tau-\phi) |\hat q(\phi-\xi)|\, d\sigma(\xi) d\sigma(\tau) d\phi.$$

The proof of Lemma 3.2 follows the lines of the proof of Proposition 3. Heuristically, Lemma 3.2 is derived from Proposition 3 replacing the domain $\Gamma(\eta)$ for the variable ϕ by the tubular neighborhood $\Gamma_\infty(\eta)$ which is the result of widening the sphere $\Gamma(\eta)$ a distance 1 in the normal direction. Nevertheless there is an additional difficulty which has to be managed: the fact that neither f_3 nor f_4 are compactly supported and their Fourier transform can not be controlled by the maximal operator using Lemma 6.2. But we must keep in mind that we can apply Lemma 6.2 to two functions, f_1, f_2, which are compactly supported, and the integral of $|\widehat{f_3}|^2$ or $|\widehat{f_4}|^2$ in ϕ can be bounded by the L^2-norm using that the variable ϕ is solid. After these comments, we omit the long and tedious proof of Lemma 3.2. The reader can see all the details in a similar situation for the cubic term in 2D (Lemma 2.2.3 of [25]).

The key to control the principal value term (3.3) remains in the following lemma whose proof follows the lines of the proof of Proposition 3.

Lemma 3.4. *We denote*

$$\mathcal{F}(Q_{\delta_1,\delta_2,\delta_3}(q))(\eta) := 1/|\eta|^6 \, \chi_{(\delta_1^{-1},\infty)}(|\eta|)\, \chi_{(\delta_2^{-1},\infty)}(|\eta|)\, \chi_{(\delta_3^{-1},\infty)}(|\eta|) \times \int_{\Gamma_{\delta_1}(\eta)} \int_{\Gamma_{\delta_2}(\eta)} \int_{\Gamma_{\delta_3}(\eta)} |\hat{q}(\xi)\hat{q}(\eta-\tau)\hat{q}(\tau-\phi)\hat{q}(\phi-\xi)|\, d\xi d\tau d\phi.$$

Let q and α as in Lemma 3.1 and $\beta < \alpha + 1/2$, $\varepsilon = \alpha + 1/2 - \beta > 0$. Then there exist $\delta_0 > 0$ and $\gamma^* = \gamma^*(\epsilon) > 1$ such that for all $\delta_1, \delta_2, \delta_3$ satisfying $0 < \delta_1, \delta_2, \delta_3 \le \delta_0$ it holds

$$\|Q_{\delta_1,\delta_2,\delta_3}(q)\|_{\dot{W}^{\beta,2}} \le C(\delta_1\delta_2\delta_3)^{\gamma^*} \, \|q\|^4_{W^{\alpha,2}(\mathbb{R}^3)},$$

where C only depends on α, β, δ_0 and the support of q.

Analogously to the comment about Lemma 3.1 above, this result should not be surprising since $\mathcal{F}Q_{\delta_1,\delta_2,\delta_3}(q)(\eta) \sim \delta_1\delta_2\delta_3\mathcal{F}Q(q)(\eta)$, for $\delta_1, \delta_2, \delta_3$ small. To estimate the term (3.3) we have to take the partition (3.15) of $\mathbb{R}^3$ with j_1 the lowest integer such that $j_1 \ge 1 - \log_2(\delta_0)$, for δ_0 from Lemma 3.4. In particular, the control of the ring terms

$$\int_{\Gamma_j(\eta)} \int_{\Gamma_k(\eta)} \int_{\Gamma_l(\eta)} \frac{\hat{q}(\xi)\hat{q}(\eta-\tau)\hat{q}(\tau-\phi)\hat{q}(\phi-\xi)}{[\xi\cdot(\eta-\xi)][\tau\cdot(\eta-\tau)][\phi\cdot(\eta-\phi)]}\, d\xi d\tau d\phi\,, \qquad j_1 \le j,k,l \le [\log_2|\eta|]$$

follows from Lemma 3.4 with $\delta_1 = 2^{-j+1}, \delta_2 = 2^{-k+1}, \delta_3 = 2^{-l+1}$, together with the fact that

$$|\xi\cdot(\eta-\xi)| = (|\xi - \frac{\eta}{2}| + |\frac{\eta}{2}|)(|\xi - \frac{\eta}{2}| - |\frac{\eta}{2}|) \ge c|\eta|^2\delta_1,$$

where $c > 0$ and we use definition (3.12).

To estimate the central term

$$\int_{\Gamma^*_\infty(\eta)} \int_{\Gamma^*_\infty(\eta)} \int_{\Gamma^*_\infty(\eta)} \frac{\hat{q}(\xi)\hat{q}(\eta-\tau)\hat{q}(\tau-\phi)\hat{q}(\phi-\xi)}{[\xi\cdot(\eta-\xi)][\tau\cdot(\eta-\tau)][\phi\cdot(\eta-\phi)]}\, d\xi d\tau d\phi$$

we must replace each integral on the ring $\Gamma^*_\infty(\eta)$ by $\int_{\Gamma^+_\varepsilon(\eta)} + \int_{\Gamma^-_\varepsilon(\eta)}$. We, then, use the map $F : \Gamma^+_\varepsilon(\eta) \to \Gamma^-_\varepsilon(\eta)$. We need again Calderón estimate for first differences and its analogous estimate for second differences:

Lemma 3.5. *Let $u \in W^{2,p}(\mathbb{R}^n)$, $p > 1$, $a, b, c \in \mathbb{R}^n$. Then*

$$|u(x-a) + u(x+b) - u(x) - u(x+b-a)| \le C\,|a|\,|b| \sum_{j=1}^{4} M^2|D^2u|(x_j),$$

where D^2u denotes the matrix of derivatives of order two and $x_1 = x$, $x_2 = x - a$, $x_3 = x + b$, $x_4 = x + b - a$.

These tools allow us to reduce to a sum of integrals, analogous to those written after Lemma 3.3 for the case (3.7).

3.1. **Proof of Proposition 3.**

Let us split the set $\Gamma(\eta)^3$ into the following regions

$$I(\eta) := \left\{ (\xi,\tau,\phi) \in \Gamma(\eta)^3 \,:\, |\phi - \xi| \ge \frac{|\eta|}{100}\,,\, |\phi - \tau| \ge \frac{|\eta|}{100} \right\}\,,$$

$$II(\eta) := \left\{ (\xi,\tau,\phi) \in \Gamma(\eta)^3 \,:\, |\phi - \xi| \ge \frac{|\eta|}{100}\,,\, |\phi - \tau| \le \frac{|\eta|}{100} \right\}\,,$$

$$III(\eta) := \left\{ (\xi,\tau,\phi) \in \Gamma(\eta)^3 \ : \ |\phi-\xi| \le \frac{|\eta|}{100} \, , \, |\phi-\tau| \ge \frac{|\eta|}{100} \right\} ,$$
$$IV(\eta) := \left\{ (\xi,\tau,\phi) \in \Gamma(\eta)^3 \ : \ |\phi-\xi| \le \frac{|\eta|}{100} \, , \, |\phi-\tau| \le \frac{|\eta|}{100} \right\} .$$

In this way, we can write $Q(q) = Q_I(q) + Q_{II}(q) + Q_{III}(q) + Q_{IV}(q)$. We will prove that

$$\|Q_I(q)\|_{\dot W^{\beta,2}} \le C \left(\|q\|_{L^2}^3 + \|q\|_{\dot W^{-\frac12,2}} \|q\|_{L^2}^2 \right) \|q\|_{\dot W^{\beta-1/2\,,\,2}} \, , \tag{3.23}$$
$$\|Q_{II}(q)\|_{\dot W^{\beta,2}} \le C \left(\|q\|_{L^2}^3 + \|q\|_{\dot W^{-\frac12,2}} \|q\|_{L^2}^2 \right) \|q\|_{\dot W^{\beta-1/2\,,\,2}} \, , \tag{3.24}$$
$$\|Q_{IV}(q)\|_{\dot W^{\beta,2}} \le C \left(\|q\|_{\dot W^{-\frac12,2}} \|q\|_{\dot W^{-\varepsilon,2}} \|q\|_{L^2} + \|q\|_{L^2}^2 \|q\|_{\dot W^{-\frac12-\varepsilon\,,\,2}} \right) \|q\|_{\dot W^{\beta-\frac12+\varepsilon\,,\,2}} \, , \tag{3.25}$$

provided that $\varepsilon > 0$. Note that $Q_{III}(q)$ satisfies the estimate (3.24) since $Q_{II}(q) = Q_{III}(q)$.

Proof of estimate (3.24)**.** Taking the change of variable $\phi = \eta - \phi'$, we have

$$\widehat{Q_{II}(q)}(\eta) = \frac{1}{|\eta|^3} \int\int\int_{II(\eta)} |\hat q(\xi) \hat q(\eta-\tau) \hat q(\tau-\phi') \hat q(\phi'-\xi)| \, d\sigma(\xi) d\sigma(\tau) d\sigma(\phi')$$
$$= \frac{1}{|\eta|^3} \int\int\int_{\{(\xi,\tau,\phi):(\xi,\tau,\eta-\phi)\in II(\eta)\}} |\hat q(\xi) \hat q(\eta-\tau) \hat q(\tau+\phi-\eta) \hat q(\eta-\phi-\xi)| \, d\sigma(\xi) d\sigma(\tau) d\sigma(\phi) .$$

We decompose

$$\{(\xi,\tau,\phi) \in \Gamma(\eta)^3 \ : \ (\xi,\tau,\eta-\phi) \in II(\eta)\} = \bigcup_{k=1}^{\infty} (II_k(\eta) \cup \widetilde{II}_k(\eta)),$$

where for any $k \in \mathbb{N}$, we denote

$$II_k(\eta) := \left\{ |\eta-\tau-\phi| \le \frac{|\eta|}{100} \, , \, |\eta-\xi-\phi| \ge \frac{|\eta|}{100} \, , \, |\phi-\xi| \sim 2^{-k}|\eta| \, , \, |\phi| \le |\xi| \right\} ,$$
$$\widetilde{II}_k(\eta) := \left\{ |\eta-\tau-\phi| \le \frac{|\eta|}{100} \, , \, |\eta-\xi-\phi| \ge \frac{|\eta|}{100} \, , \, |\phi-\xi| \sim 2^{-k}|\eta| \, , \, |\xi| \le |\phi| \right\} ,$$

with $(\xi,\tau,\phi) \in \Gamma(\eta)^3$. We have

$$\widehat{Q_{II}(q)}(\eta) \le \sum_{k=1}^{+\infty} \left(\widehat{Q_{II_k}(q)}(\eta) + \widehat{Q_{\widetilde{II}_k}(q)}(\eta) \right) ,$$

and then to prove (3.24) we use

$$\|Q_{II}(q)\|_{\dot W^{\beta,2}} \le \sum_{k=1}^{+\infty} \left(\|Q_{II_k}(q)\|_{\dot W^{\beta,2}} + \|Q_{\widetilde{II}_k}(q)\|_{\dot W^{\beta,2}} \right) .$$

For each $k \ge 1$ we claim

$$\|Q_{II_k}(q)\|_{\dot W^{\beta,2}} \le C \, 2^{-k/2} \|q\|_{L^2}^3 \|q\|_{\dot W^{\beta-\frac12\,,\,2}} \, , \tag{3.24a}$$
$$\|Q_{\widetilde{II}_k}(q)\|_{\dot W^{\beta,2}} \le C \, 2^{-k/2} \left(\|q\|_{L^2} + \|q\|_{\dot W^{-\frac12,2}} \right) \|q\|_{L^2}^2 \|q\|_{\dot W^{\beta-\frac12\,,\,2}} \, . \tag{3.24b}$$

In the following, we use the notation in Lemma 6.3, which is the key of the proof of the above claims.

Proof of claim (3.24a)**.** By Cauchy-Schwartz inequality,

$$\widehat{Q_{II_k}(q)}(\eta) \le \frac{1}{|\eta|^3} \left(\int\int\int_{II_k(\eta)} |\hat q(\xi) \hat q(\tau+\phi-\eta)|^2 d\sigma(\xi) d\sigma(\tau) d\sigma(\phi) \right)^{\frac12} \tag{3.26}$$

$$\times \left(\int \int \int_{II_k(\eta)} |\hat{q}(\eta - \tau')\hat{q}(\eta - \phi' - \xi')|^2 d\sigma(\xi') d\sigma(\tau') d\sigma(\phi') \right)^{\frac{1}{2}} . \tag{3.27}$$

If we widen the sphere $\Gamma(\eta)$ until $\Gamma_1(\eta) := \left\{ x \in \mathbb{R}^3 : \left| |x - \frac{\eta}{2}| - \frac{|\eta|}{2} \right| < 1 \right\}$, by part (1) of Lemma 6.2 we have

$$\begin{aligned} \int \int_{\Gamma(\eta)\times\Gamma(\eta)} |\hat{q}(\tau + \phi - \eta)|^2 d\sigma(\tau) d\sigma(\phi) &\leq C \int_{\Gamma(\eta)} \int_{\Gamma_1(\eta)} M\hat{q}(x + \phi - \eta)^2 dx\, d\sigma(\phi) \\ &\leq C\, \sigma(\Gamma(\eta)) \|M\hat{q}\|_{L^2}^2 \leq C\, |\eta|^2 \|q\|_{L^2}^2 , \end{aligned}$$

where the last inequality follows from the boundedness of Hardy-Littlewood maximal operator in $L^2(\mathbb{R}^3)$ and Plancherel identity, since the measure of $\Gamma(\eta)$ is $\pi|\eta|^2$. In the same way,

$$\int_{\Gamma(\eta)} |\hat{q}(\eta - \tau')|^2 d\sigma(\tau') \leq C\, \|q\|_{L^2}^2 . \tag{3.28}$$

If $(\xi, \tau, \phi) \in II_k(\eta)$ then $|\xi| \geq 2^{-k-2}|\eta|$, and changing the order of integration in ξ and η by Lemma 6.1, it holds

$$\begin{aligned} \|Q_{II_k}(q)\|_{\dot{W}^{\beta,2}}^2 &\leq C\, \|q\|_{L^2}^4 \int_{\mathbb{R}^3} |\eta|^{2\beta-4} \int_{\{\xi\in\Gamma(\eta):|\xi|\geq 2^{-k-2}|\eta|\}} |\hat{q}(\xi)|^2 \\ &\quad \times \int \int_{A_k(\eta)} |\hat{q}(\eta - \phi' - \xi')|^2 d\sigma(\xi') d\sigma(\phi') d\sigma(\xi) d\eta \\ &= C\, \|q\|_{L^2}^4 \int_{\mathbb{R}^3} |\hat{q}(\xi)|^2 \int_{\{\eta\in\Lambda(\xi)\,:\,|\xi|\geq 2^{-k-2}|\eta|\}} \frac{|\eta|}{|\xi|}\, |\eta|^{2\beta-4} \\ &\quad \times \int \int_{A_k(\eta)} |\hat{q}(\eta - \phi' - \xi')|^2 d\sigma(\xi') d\sigma(\phi') d\sigma(\eta) d\xi \\ &\leq C\, 2^k \|q\|_{L^2}^4 \int_{\mathbb{R}^3} |\hat{q}(\xi)|^2 F_k(\xi)\, d\xi \leq C\, 2^{-k} \|q\|_{L^2}^6 \|q\|_{\dot{W}^{\beta-\frac{1}{2},2}}^2 , \end{aligned} \tag{3.29}$$

where the last inequality follows from part (i) of Lemma 6.3 and $F_k(\xi)$ is defined in (6.1). Also,

$$A_k(\eta) := \left\{ (\xi', \phi') \in \Gamma(\eta) \times \Gamma(\eta) \,:\, |\xi' - \phi'| \leq 2^{-k+1}|\eta| \,,\, |\eta - \phi' - \xi'| \geq \frac{|\eta|}{100} \right\} . \tag{3.30}$$

□

Proof of claim (3.24b)**.** We take $\widetilde{II}_k(\eta) = \widetilde{II}_k^1(\eta) \cup \widetilde{II}_k^2(\eta)$, where

$$\begin{aligned} \widetilde{II}_k^1(\eta) &:= \left\{ (\xi, \tau, \phi) \in \widetilde{II}_k(\eta) \,:\, |\eta - \phi - \tau| \leq 2^{-k-3}|\eta| \right\} , \\ \widetilde{II}_k^2(\eta) &:= \left\{ (\xi, \tau, \phi) \in \widetilde{II}_k(\eta) \,:\, |\eta - \phi - \tau| \geq 2^{-k-3}|\eta| \right\} . \end{aligned}$$

Let us start with the domain

$$\widetilde{II}_k^1(\eta) = \{(\xi, \tau, \phi) \in \Gamma(\eta)^3 : |\eta - \phi - \tau| \leq 2^{-k-3}|\eta|, |\eta - \xi - \tau| \geq |\eta|/100, |\phi - \xi| \sim 2^{-k}|\eta|, |\xi| \leq |\phi|\}.$$

On this region $|\eta - \tau| \geq 2^{-k-3}|\eta|$ holds. By Cauchy-Schwartz inequality,

$$\widehat{Q_{\widetilde{II}_k^1}(q)}(\eta) \leq \frac{1}{|\eta|^3} \left(\int \int \int_{\widetilde{II}_k^1(\eta)} |\hat{q}(\xi)\hat{q}(\eta - \tau)|^2 d\sigma(\xi) d\sigma(\tau) d\sigma(\phi) \right)^{\frac{1}{2}} \tag{3.31}$$

$$\times \left(\int \int \int_{\widetilde{II}_k^1(\eta)} |\hat{q}(\tau' + \phi' - \eta)\hat{q}(\eta - \phi' - \xi')|^2 d\sigma(\xi') d\sigma(\tau') d\sigma(\phi') \right)^{\frac{1}{2}} . \tag{3.32}$$

By part (1) of Lemma 6.2, we have $\int_{\Gamma(\eta)} |\hat{q}(\xi)|^2 d\sigma(\xi) \le C\|q\|_{L^2}^2$, and by this lemma and Fubini's theorem, for each $\xi', \phi' \in \Gamma(\eta)$ in (3.32),

$$\int_{\Gamma(\eta)} |\hat{q}(\tau' + \phi' - \eta)|^2 d\sigma(\tau') \le C\|q\|_{L^2}^2 .$$

Taking the change $\zeta = \eta - \tau$, and changing the order of integration in ζ and η by Lemma 6.1, we may write

$$\begin{aligned}
\|Q_{\widetilde{II}_k^1}(q)\|_{\dot{W}^{\beta,2}}^2 &\le C\|q\|_{L^2}^4 \int_{\mathbb{R}^3} |\eta|^{2\beta-4} \int_{\{\zeta\in\Gamma(\eta):\,|\zeta|\ge 2^{-k-3}|\eta|\}} |\hat{q}(\zeta)|^2 d\sigma(\zeta) \\
&\quad \times \int\int_{A_k(\eta)} |\hat{q}(\eta-\phi'-\xi')|^2 d\sigma(\xi') d\sigma(\phi') d\eta \\
&= C\|q\|_{L^2}^4 \int_{\mathbb{R}^3} |\hat{q}(\zeta)|^2 \int_{\{\eta\in\Lambda(\zeta):\,|\zeta|\ge 2^{-k-3}|\eta|\}} \frac{|\eta|}{|\zeta|} |\eta|^{2\beta-4} \\
&\quad \times \int\int_{A_k(\eta)} |\hat{q}(\eta-\phi'-\xi')|^2 d\sigma(\xi') d\sigma(\phi') d\sigma(\eta) d\zeta \\
&\le C\|q\|_{L^2}^4\, 2^k \int_{\mathbb{R}^3} |\hat{q}(\zeta)|^2 F_k(\zeta)\, d\zeta \le C\, 2^{-k} \|q\|_{L^2}^6 \|q\|_{\dot{W}^{\beta-\frac{1}{2},2}}^2 ,
\end{aligned}$$

where the last inequality follows from part (i) of Lemma 6.3, and $A_k(\eta)$, $F_k(\zeta)$ are defined in (3.30), (6.1).

We go on with the region $\widetilde{II}_k^2(\eta)$. Let us split it as follows: $\widetilde{II}_k^2(\eta) = \widetilde{II}_{k,a}^2(\eta) \cup \widetilde{II}_{k,b}^2(\eta)$, where

$$\begin{aligned}
\widetilde{II}_{k,a}^2(\eta) &:= \left\{ (\xi,\tau,\phi) \in \widetilde{II}_k^2(\eta) \,:\, |\eta-\tau| \le |\phi| \right\} , \\
\widetilde{II}_{k,b}^2(\eta) &:= \left\{ (\xi,\tau,\phi) \in \widetilde{II}_k^2(\eta) \,:\, |\eta-\tau| \ge |\phi| \right\} .
\end{aligned}$$

On the region $\widetilde{II}_{k,a}^2(\eta)$, we know that if $|\xi| \ge 2^{-k}|\eta|$ we can follow the lines of the case $II_k(\eta)$. So, splitting once more as $\widetilde{II}_{k,a}^2(\eta) = \widetilde{II}_{k,a,1}^2(\eta) \cup \widetilde{II}_{k,a,2}^2(\eta)$, where

$$\begin{aligned}
\widetilde{II}_{k,a,1}^2(\eta) &:= \left\{ (\xi,\tau,\phi) \in \widetilde{II}_{k,a}^2(\eta) \,:\, |\xi| \ge 2^{-k}|\eta| \right\} \\
&= \{(\xi,\tau,\phi)\in\Gamma(\eta)^3 : 2^{-k-3}|\eta| \le |\eta-\phi-\tau| \le |\eta|/100,\, |\eta-\xi-\tau| \ge |\eta|/100, \\
&\qquad |\phi-\xi| \sim 2^{-k}|\eta|,\, 2^{-k}|\eta| \le |\xi| \le |\phi|,\, |\eta-\tau| \le |\phi|\} \\
\widetilde{II}_{k,a,2}^2(\eta) &:= \left\{ (\xi,\tau,\phi) \in \widetilde{II}_{k,a}^2(\eta) \,:\, |\xi| \le 2^{-k}|\eta| \right\} \\
&= \{(\xi,\tau,\phi)\in\Gamma(\eta)^3 : 2^{-k-3}|\eta| \le |\eta-\phi-\tau| \le |\eta|/100,\, |\eta-\xi-\tau| \ge |\eta|/100, \\
&\qquad |\phi-\xi| \sim 2^{-k}|\eta|,\, |\xi| \le 2^{-k}|\eta|,\, |\xi| \le |\phi|,\, |\eta-\tau| \le |\phi|\}.
\end{aligned}$$

We may write

$$\|Q_{\widetilde{II}_{k,a,1}^2}(q)\|_{\dot{W}^{\beta,2}} \le C\, 2^{-k/2} \|q\|_{L^2}^3 \|q\|_{\dot{W}^{\beta-\frac{1}{2},2}} . \tag{3.33}$$

In this way, we reduce to the case $\widetilde{II}_{k,a,2}^2(\eta)$, where $|\phi| \le 3\cdot 2^{-k}|\eta|$ holds.

Remark. From the proof of claim 6 in [28] one deduces to the following estimates:

$$\int\int_{\Gamma(\eta)^2\cap\{|x|\le|y|\le 2^{-k+1}|\eta|,\,|x-y|\le\frac{|\eta|}{100}\}} |\hat{q}(x-y)|^2 d\sigma(y) d\sigma(x) \le C\, 2^{-k} |\eta|^2 \|q\|_{\dot{W}^{-\frac{1}{2},2}}^2, \tag{3.34}$$

$$\int\int_{\Gamma(\eta)^2\cap\{|x|\le|y|\le 2^{-k+1}|\eta|,\, 2^{-k-3}|\eta|\le|x-y|\le\frac{|\eta|}{100}\}} |\hat{q}(x-y)|^2 d\sigma(y) d\sigma(x) \le C\, |\eta|\, \|q\|_{L^2}^2. \tag{3.35}$$

By Cauchy-Schwartz inequality as in (3.26),(3.27), the fact that Lemma 6.2 implies (3.28), applying the change $\zeta = \eta - \tau$, since

$$\int\int_{\Gamma(\eta)^2 \cap \{|\zeta| \le |\phi| \le 3\cdot 2^{-k}|\eta|,\, 2^{-k-3}|\eta| \le |\zeta-\phi| \le \frac{|\eta|}{100}\}} |\hat{q}(\phi-\zeta)|^2 d\sigma(\zeta) d\sigma(\phi) \le C\,|\eta|\,\|q\|^2_{L^2}$$

by (3.35), and changing the order of integration in ξ and η by Lemma 6.1, we have

$$\|Q_{\widetilde{II}^2_{k,a,2}}(q)\|^2_{\dot W^{\beta,2}} \le C\|q\|^4_{L^2} \int_{\mathbb{R}^3} |\eta|^{2\beta-5} \int_{\Gamma(\eta)} |\hat{q}(\xi)|^2 d\sigma(\xi) \tag{3.36}$$

$$\times \int\int_{A_k(\eta)} |\hat{q}(\eta-\phi'-\xi')|^2 d\sigma(\xi') d\sigma(\phi') d\eta \tag{3.37}$$

$$\le C\|q\|^4_{L^2} \int_{\mathbb{R}^3} \frac{|\hat{q}(\xi)|^2}{|\xi|} F_k(\xi)\, d\xi \tag{3.38}$$

$$\le C\, 2^{-2k} \|q\|^2_{\dot W^{-\frac12,2}} \|q\|^4_{L^2} \|q\|^2_{\dot W^{\beta-\frac12},2}\,,$$

where the last inequality follows from part (i) of Lemma 6.3 and $A_k(\eta)$, $F_k(\xi)$ are defined in (3.30), (6.1).

Let us go on with the domain

$$\widetilde{II}^2_{k,b}(\eta) = \{(\xi,\tau,\phi) \in \Gamma(\eta)^3 : 2^{-k-3}|\eta| \le |\eta-\phi-\tau| \le |\eta|/100,$$
$$|\eta-\xi-\tau| \ge |\eta|/100,\, |\phi-\xi| \sim 2^{-k}|\eta|,\, |\xi| \le |\phi| \le |\eta-\tau|\}.$$

So, if $(\xi,\tau,\phi) \in \widetilde{II}^2_{k,b}(\eta)$ then $|\eta-\tau| \ge 2^{-k-4}|\eta|$. Following the lines for the case $\widetilde{II}^1_k(\eta)$, one obtains

$$\|Q_{\widetilde{II}^2_{k,b}}(q)\|_{\dot W^{\beta,2}} \le C\, 2^{-k/2} \|q\|^3_{L^2} \|q\|_{\dot W^{\beta-\frac12},2}\,. \tag{3.39}$$

We conclude the estimate

$$\|Q_{\widetilde{II}^2_{k}}(q)\|_{\dot W^{\beta,2}} \le C\, 2^{-k/2} \left(\|q\|_{L^2} + \|q\|_{\dot W^{-\frac12,2}} \right) \|q\|^2_{L^2} \|q\|_{\dot W^{\beta-\frac12},2}\,, \tag{3.40}$$

and the claim (3.24b).

This ends the proof of estimate (3.24).

□

Proof of estimate (3.23) **.** Taking the change of variable $\phi = \eta - \phi'$, we have

$$\widehat{Q_I(q)}(\eta) = \frac{1}{|\eta|^3} \int\int\int_{I(\eta)} |\hat{q}(\xi)\hat{q}(\eta-\tau)\hat{q}(\tau-\phi')\hat{q}(\phi'-\xi)|\, d\sigma(\xi) d\sigma(\tau) d\sigma(\phi')$$
$$= \frac{1}{|\eta|^3} \int\int\int_{\{(\xi,\tau,\eta-\phi)\in I(\eta)\}} |\hat{q}(\xi)\hat{q}(\eta-\tau)\hat{q}(\tau+\phi-\eta)\hat{q}(\eta-\phi-\xi)|\, d\sigma(\xi) d\sigma(\tau) d\sigma(\phi)\,.$$

For $\eta \in \mathbb{R}^3$ fixed, we take the decomposition

$$\left\{ (\xi,\tau,\phi) \in \Gamma(\eta)^3 \;:\; |\eta-\phi-\xi| \ge \frac{|\eta|}{100}, |\eta-\phi-\tau| \ge \frac{|\eta|}{100} \right\} = I_1(\eta) \cup \widetilde{I}_1(\eta) \cup I_2(\eta)\,,$$

where

$$I_1(\eta) := \left\{ |\phi-\tau| \le \frac{|\eta|}{400}\,,\, |\phi| \le |\eta-\tau|\,,\, |\eta-\tau-\phi| \ge \frac{|\eta|}{100}\,,\, |\eta-\xi-\phi| \ge \frac{|\eta|}{100} \right\}\,,$$
$$\widetilde{I}_1(\eta) := \left\{ |\phi-\tau| \le \frac{|\eta|}{400}\,,\, |\phi| \ge |\eta-\tau|\,,\, |\eta-\tau-\phi| \ge \frac{|\eta|}{100}\,,\, |\eta-\xi-\phi| \ge \frac{|\eta|}{100} \right\}\,,$$

$$I_2(\eta) := \left\{ |\phi - \tau| \geq \frac{|\eta|}{400} \,,\, |\eta - \tau - \phi| \geq \frac{|\eta|}{100} \,,\, |\eta - \xi - \phi| \geq \frac{|\eta|}{100} \right\} ,$$

with $(\xi, \tau, \phi) \in \Gamma(\eta)^3$. It holds

$$\widehat{Q_I(q)}(\eta) = \widehat{Q_{I_1}(q)}(\eta) + \widehat{Q_{\widetilde{I}_1}(q)}(\eta) + \widehat{Q_{I_2}(q)}(\eta) \,,$$

and also,

$$\|Q_I(q)\|_{\dot W^{\beta,2}} \leq \|Q_{I_1}(q)\|_{\dot W^{\beta,2}} + \|Q_{\widetilde{I}_1}(q)\|_{\dot W^{\beta,2}} + \|Q_{I_2}(q)\|_{\dot W^{\beta,2}} \,.$$

We claim the following:

$$\|Q_{I_1}(q)\|_{\dot W^{\beta,2}} \leq C\, \|q\|_{L^2}^3 \|q\|_{\dot W^{\beta-\frac{1}{2},2}} \,, \tag{3.23a}$$

$$\|Q_{\widetilde{I}_1}(q)\|_{\dot W^{\beta,2}} \leq C\, \|q\|_{L^2}^3 \|q\|_{\dot W^{\beta-\frac{1}{2},2}} \,, \tag{3.23b}$$

$$\|Q_{I_2}(q)\|_{\dot W^{\beta,2}} \leq C\, \|q\|_{\dot W^{-\frac{1}{2},2}} \|q\|_{L^2}^2 \|q\|_{\dot W^{\beta-\frac{1}{2},2}} \,. \tag{3.23c}$$

The estimate (3.23) follows from these three claims. In their proofs we use the notation introduced in the key Lemma 6.3 located in the appendix.

□

Proof of claim (3.23a) **.** On this region we have $|\eta - \tau| \geq \frac{|\eta|}{200}$. Applying the Cauchy-Schwartz inequality as in (3.31)-(3.32), since for $\eta \in \mathbb{R}^3$, $\phi' \in \Gamma(\eta)$ fixed, by Lemma 6.2 it holds $\int_{\Gamma(\eta)} |\hat q(\xi)|^2 d\sigma(\xi) \leq C\, \|q\|_{L^2}^2$, $\int_{\Gamma(\eta)} |\hat q(\eta - \phi' - \xi')|^2 d\sigma(\xi') \leq C\, \|q\|_{L^2}^2$ and $\sigma(\Gamma(\eta)) = \pi |\eta|^2$, we obtain

$$\begin{aligned} \widehat{Q_{I_1}(q)}(\eta) \leq \frac{C}{|\eta|^2}\, \|q\|_{L^2}^2 \left(\int_{\{\tau\in\Gamma(\eta) : |\eta-\tau| \geq \frac{|\eta|}{200}\}} |\hat q(\eta - \tau)|^2 d\sigma(\tau) \right)^{\frac{1}{2}} \\ \times \left(\int\int_{A(\eta)} |\hat q(\tau' + \phi' - \eta)|^2 d\sigma(\tau') d\sigma(\phi') \right)^{\frac{1}{2}} , \end{aligned}$$

where

$$A(\eta) := \left\{ (\tau', \phi') \in \Gamma(\eta)^2 \,:\, |\tau' - \phi'| \leq \frac{|\eta|}{400} \,,\, |\eta - \tau' - \phi'| \geq \frac{|\eta|}{100} \right\} . \tag{3.41}$$

Taking $\zeta = \eta - \tau$ and changing the order of integration in ζ and η by Lemma 6.1, we have

$$\begin{aligned} \|Q_{I_1}(q)\|_{\dot W^{\beta,2}}^2 &\leq C \|q\|_{L^2}^4 \int_{\mathbb{R}^3} |\hat q(\zeta)|^2 \int_{\{\eta\in\Lambda(\zeta) :\, |\zeta| \geq \frac{|\eta|}{200}\}} \frac{|\eta|}{|\zeta|}\, |\eta|^{2\beta-4} \\ &\quad \times \int\int_{A(\eta)} |\hat q(\tau' + \phi' - \eta)|^2 d\sigma(\tau') d\sigma(\phi') d\sigma(\eta) d\zeta \\ &\leq C \|q\|_{L^2}^4 \int_{\mathbb{R}^3} |\hat q(\zeta)|^2 F_1(\zeta)\, d\zeta \leq C \|q\|_{L^2}^6 \|q\|_{\dot W^{\beta-\frac{1}{2},2}}^2 , \end{aligned}$$

where the last inequality follows from part (i) of Lemma 6.3 with $k = 1$ and $F_k(\zeta)$ is defined in (6.1). Let us remark that $A(\eta) \subset A_1(\eta)$ according to the notation in (3.30).

□

Proof of claim (3.23b) **.** We consider the partition $\widetilde{I}_1(\eta) = \widetilde{I}_{1,a}(\eta) \cup \widetilde{I}_{1,b}(\eta)$, where

$$\widetilde{I}_{1,a}(\eta) := \left\{ (\xi, \tau, \phi) \in \widetilde{I}_1(\eta) \,:\, |\eta - \phi| \leq \frac{|\eta|}{200} \right\} ,$$

$$\widetilde{I}_{1,b}(\eta) := \left\{ (\xi,\tau,\phi) \in \widetilde{I}_1(\eta) \,:\, |\eta-\phi| \geq \frac{|\eta|}{200} \right\} .$$

On the region $\widetilde{I}_{1,a}(\eta)$, $|\xi| \geq \frac{|\eta|}{200}$ holds. We may follow the same lines as in the proof for the case $II_k(\eta)$ with $k=1$, interchanging the roles of the factors $\hat{q}(\eta-\phi-\xi)$ and $\hat{q}(\tau+\phi-\eta)$. As we mentioned before, $A(\eta) \subset A_1(\eta)$, according to the notation in (3.30) and we may apply part (i) of Lemma 6.3 for $k=1$. We have

$$\|Q_{\widetilde{I}_{1,a}}(q)\|_{\dot{W}^{\beta,2}} \leq C\|q\|_{L^2}^3 \|q\|_{\dot{W}^{\beta-\frac{1}{2},2}} .$$

On the region $\widetilde{I}_{1,b}(\eta)$, $|\eta-\tau| \geq \frac{|\eta|}{400}$ holds. Hence, the proof of the estimate for the region $I_1(\eta)$ is valid here, deducing

$$\|Q_{\widetilde{I}_{1,b}}\|_{\dot{W}^{\beta,2}} \leq C\|q\|_{L^2}^3 \|q\|_{\dot{W}^{\beta-\frac{1}{2},2}} .$$

□

Proof of claim (3.23c) **.**

Let $\eta \in \mathbb{R}^3 \setminus \{0\}$. The occurrences of q in the term $\widehat{Q_{I_2}(q)}(\eta)$ interact with each other, so that we only may bound by the maximal operator just once. We need to consider an extra splitting carried out by taking the set of points $\{\theta_j \,:\, 1 \leq j \leq \mathcal{N}\}$ in the unitary sphere $\mathbb{S}^2$, where $\mathcal{N}$ is large enough, to get a covering of the sphere $\Gamma(\eta)$ with $\mathcal{N}$ spherical cups $J_j(\eta)$ centered at Ω_j of radius $C_1|\eta|$, for a certain constant $C_1 > 0$ to be chosen later (with $\mathcal{N} \sim \frac{1}{C_1^2}$). We define for every j

$$\Omega_j = \frac{\eta}{2} + \frac{|\eta|}{2}\,\theta_j . \tag{3.42}$$

Then

$$\Gamma(\eta) = \bigcup_{j=1}^{\mathcal{N}} J_j(\eta) , \quad \widehat{Q_{I_2}(q)}(\eta) \leq \sum_{j=1}^{\mathcal{N}} \widehat{R_{J_j}(q)}(\eta) ,$$

where

$$\begin{aligned}\widehat{R_{J_j}(q)}(\eta) := \frac{1}{|\eta|^3} \int_{J_j(\eta)} \int_{X_\eta(\tau)} \int_{Y_\eta(\phi)} &|\hat{q}(\xi)\hat{q}(\eta-\tau)\hat{q}(\tau+\phi-\eta)| \\ &\times |\hat{q}(\eta-\phi-\xi)|\, d\sigma(\xi) d\sigma(\phi) d\sigma(\tau),\end{aligned}$$

and

$$X_\eta(\tau) := \{\phi \in \Gamma(\eta) \,:\, |\phi-\tau| \geq |\eta|/400 \,,\, |\eta-\phi-\tau| \geq |\eta|/100\} , \tag{3.43}$$

$$Y_\eta(\phi) := \{\xi \in \Gamma(\eta) \,:\, |\eta-\phi-\xi| \geq |\eta|/100\} . \tag{3.44}$$

We fix $j \in \{1, \dots, \mathcal{N}\}$. In this part, we take an orthonormal reference of $\mathbb{R}^3$ $\{e_1, e_2, e_3\}$ such that $e_1 = \theta_j$, according to the notation used in (3.42). On the integral expression $\widehat{R_{J_j}(q)}(\eta)$, we apply Cauchy-Schwartz inequality as in (3.26)-(3.27). For each j, η fixed, by Fubini's theorem, we have

$$\begin{aligned}&\int_{J_j(\eta)} \int_{X_\eta(\tau)} \int_{Y_\eta(\phi)} |\hat{q}(\xi)\hat{q}(\tau+\phi-\eta)|^2 d\sigma(\xi) d\sigma(\phi) d\sigma(\tau) \\ &\leq \int_{\Gamma(\eta)} |\hat{q}(\xi)|^2 \int_{J_j(\eta)} \int_{X_\eta(\tau)} |\hat{q}(\tau+\phi-\eta)|^2 d\sigma(\phi) d\sigma(\tau) d\sigma(\xi) .\end{aligned}$$

Moreover,

$$\begin{aligned}
&\int_{J_j(\eta)}\int_{X_\eta(\tau')}\int_{Y_\eta(\phi')} |\hat{q}(\eta-\tau')\hat{q}(\eta-\phi'-\xi')|^2 d\sigma(\xi')d\sigma(\phi')d\sigma(\tau')\\
&\le \int_{\Gamma(\eta)}\int_{\Gamma(\eta)}\int_{Y_\eta(\phi')} |\hat{q}(\eta-\tau')\hat{q}(\eta-\phi'-\xi')|^2 d\sigma(\xi')d\sigma(\phi')d\sigma(\tau')\\
&= \int_{\Gamma(\eta)} |\hat{q}(\eta-\tau')|^2 d\sigma(\tau') \int_{\Gamma(\eta)}\int_{Y_\eta(\phi')} |\hat{q}(\eta-\phi'-\xi')|^2 d\sigma(\xi')d\sigma(\phi')\\
&\le C\,\|q\|_{L^2}^2 \int_{\Gamma(\eta)}\int_{Y_\eta(\phi')} |\hat{q}(\eta-\phi'-\xi')|^2 d\sigma(\xi')d\sigma(\phi'),
\end{aligned}$$

bounding by the maximal operator using Lemma 6.2 in the last inequality. Changing the order of integration in ξ and η by Lemma 6.1, we may write

$$\begin{aligned}
\|R_{J_j}(q)\|_{\dot{W}^{\beta,2}}^2 \le C\,\|q\|_{L^2}^2 &\int_{\mathbb{R}^3} |\eta|^{2\beta-6}\int_{\Gamma(\eta)} |\hat{q}(\xi)|^2 \int_{J_j(\eta)}\int_{X_\eta(\tau)} |\hat{q}(\tau+\phi-\eta)|^2 d\sigma(\phi)d\sigma(\tau)\\
&\times \int_{\Gamma(\eta)}\int_{Y_\eta(\phi')} |\hat{q}(\eta-\phi'-\xi')|^2 d\sigma(\xi')d\sigma(\phi')d\sigma(\xi)d\eta\\
= C\,\|q\|_{L^2}^2 &\int_{\mathbb{R}^3} \frac{|\hat{q}(\xi)|^2}{|\xi|} \int_{\Lambda(\xi)} |\eta|^{2\beta-5} \int_{J_j(\eta)}\int_{X_\eta(\tau)} |\hat{q}(\tau+\phi-\eta)|^2 d\sigma(\phi)d\sigma(\tau)\\
&\times \int_{\Gamma(\eta)}\int_{Y_\eta(\phi')} |\hat{q}(\eta-\phi'-\xi')|^2 d\sigma(\xi')d\sigma(\phi')d\sigma(\eta)d\xi.
\end{aligned}$$

We fix $\xi \in \mathbb{R}^3 \setminus \{0\}$ and denote

$$\begin{aligned}
G_j(\xi) := &\int_{\Lambda(\xi)} |\eta|^{2\beta-5} \int_{J_j(\eta)}\int_{X_\eta(\tau)} |\hat{q}(\tau+\phi-\eta)|^2 d\sigma(\phi)d\sigma(\tau)\\
&\times \int_{\Gamma(\eta)}\int_{Y_\eta(\phi')} |\hat{q}(\eta-\phi'-\xi')|^2 d\sigma(\xi')d\sigma(\phi')d\sigma(\eta).
\end{aligned}$$

We write $\eta \in \Lambda(\xi)$ in cylindric coordinates $\eta = \xi + sz$, with $s \ge 0$ and $z \in \{\xi\}^\perp$, $|z| = 1$. It is true that $d\sigma(\eta) = s\,ds\,d\sigma(z)$. Let $h(s) := |\eta| = (|\xi|^2 + s^2)^{\frac{1}{2}}$. We have

$$\begin{aligned}
G_j(\xi) = &\int_{\mathbb{S}^1}\int_0^\infty h(s)^{2\beta-5} \int_{J_j(\xi+sz)}\int_{X_{\xi+sz}(\tau)} |\hat{q}(\tau+\phi-(\xi+sz))|^2 d\sigma(\phi)d\sigma(\tau)\\
&\times \int_{\Gamma(\xi+sz)}\int_{Y_{\xi+sz}(\phi')} |\hat{q}(\xi+sz-\phi'-\xi')|^2 d\sigma(\xi')d\sigma(\phi')\,s\,ds\,d\sigma(z)\\
\le C\,\|q\|_{L^2}^2 &\int_{\mathbb{S}^1}\int_0^\infty h(s)^{2\beta-4} \qquad (3.45)\\
&\times \int_{\Gamma(\xi+sz)}\int_{Y_{\xi+sz}(\phi')} |\hat{q}(\xi+sz-\phi'-\xi')|^2 d\sigma(\xi')d\sigma(\phi')\,s\,ds\,d\sigma(z)\,, \qquad (3.46)
\end{aligned}$$

where the last inequality follows from the following

Claim 3.1. *Let $1 \le j \le \mathcal{N}$, $\xi \in \mathbb{R}^3$, $z \in \{\xi\}^\perp$, with $|z| = 1$ and $s \ge 0$. Hence*

$$\begin{aligned}
&\int_{J_j(\xi+sz)}\int_{X_{\xi+sz}(\tau)} |\hat{q}(\tau+\phi-(\xi+sz))|^2 d\sigma(\phi)d\sigma(\tau) \qquad (3.47)\\
&\le C\,h(s)\|q\|_{L^2}^2.
\end{aligned}$$

Proof of claim 3.1.

We write τ, ϕ in spherical coordinates with respect to the reference $\{e_1, e_2, e_3\}$:

$$\tau = \frac{\xi + sz}{2} + \frac{h(s)}{2}\left(\sin\psi\cos\delta\, e_1 + \sin\psi\sin\delta\, e_2 + \cos\psi\, e_3\right), \tag{3.48}$$

$$\phi = \frac{\xi + sz}{2} + \frac{h(s)}{2}(\sin\zeta\cos\gamma\, e_1 + \sin\zeta\sin\gamma\, e_2 + \cos\zeta\, e_3), \tag{3.49}$$

where $\psi, \zeta \in [0, \pi]$, $\delta, \gamma \in (-\pi, \pi]$. It holds

$$d\sigma(\phi)d\sigma(\tau) = h(s)^4 \sin\psi\sin\zeta\, d\gamma\, d\zeta\, d\delta\, d\psi.$$

Notice that if $\tau \in J_j(\xi + sz)$ then τ belongs to the "curvilinear square" from the sphere $\Gamma(\eta)$ which contains the spherical cup $J_j(\xi + sz)$ given by

$$(\psi, \delta) \in \left[\frac{\pi}{2} - \varepsilon_0\,,\ \frac{\pi}{2} + \varepsilon_0\right] \times [-\varepsilon_0, \varepsilon_0],$$

where $\varepsilon_0 = \varepsilon_0(C_1)$ satisfies $\sin\varepsilon_0 = 2C_1$. For each $\zeta \in [0, \pi]$ and ψ, δ, we define

$$\begin{aligned} X^*(\zeta, \psi, \delta) &:= \{\gamma \in (-\pi, \pi] \,:\, \phi \in X_{\xi+sz}(\tau)\} \\ &= \Big\{\gamma \in (-\pi, \pi] \,:\, -(1 - \frac{1}{5000}) \le \sin\psi\cos\delta\sin\zeta\cos\gamma \\ &\qquad + \sin\psi\sin\delta\sin\zeta\sin\gamma + \cos\psi\cos\zeta \le 1 - \frac{1}{80000}\Big\}. \end{aligned}$$

The integral expression (3.47) is bounded by

$$C\int_{\frac{\pi}{2}-\varepsilon_0}^{\frac{\pi}{2}+\varepsilon_0}\int_{-\varepsilon_0}^{\varepsilon_0}\int_0^{\pi}\int_{X^*(\zeta,\psi,\delta)} h(s)^4 \sin\psi\sin\zeta\,|\hat{q}(A(j,s,\psi,\delta,\zeta,\gamma))|^2 d\gamma\, d\zeta\, d\delta\, d\psi\,,$$

where

$$\begin{aligned} A(j,s,\psi,\delta,\zeta,\gamma) := \frac{h(s)}{2}\big((\sin\psi\,\cos\delta\, +\, \sin\zeta\,\cos\gamma)\, e_1 \\ +\, (\sin\psi\,\sin\delta\, +\, \sin\zeta\,\sin\gamma)\, e_2\, +\, (\cos\psi\, +\, \cos\zeta)\, e_3\big). \end{aligned}$$

For technical reasons we divide the domain which corresponds to the angles $\psi, \delta, \zeta, \gamma$ into two pieces $\mathcal{A}_1$, $\mathcal{A}_2$:

$$\mathcal{A}_1 := \{(\psi, \delta, \zeta, \gamma) \,:\, |\cos(\psi - \zeta)| < 1 - 10^{-9}\}\,,$$
$$\mathcal{A}_2 := \{(\psi, \delta, \zeta, \gamma) \,:\, |\cos(\psi - \zeta)| \ge 1 - 10^{-9}\}\,,$$

where $(\psi, \delta, \zeta) \in \left[\frac{\pi}{2} - \varepsilon_0\,, \frac{\pi}{2} + \varepsilon_0\right] \times [-\varepsilon_0, \varepsilon_0] \times [0, \pi]$, and for each ψ, δ, ζ fixed γ belongs to the set $X^*(\zeta, \psi, \delta)$. In this way, (3.47) becomes bounded by

$$C\left(\int\int\int\int_{\mathcal{A}_1} + \int\int\int\int_{\mathcal{A}_2}\right) h(s)^4 \sin\psi\sin\zeta\,|\hat{q}(A(j,s,\psi,\delta,\zeta,\gamma))|^2 d\gamma\, d\zeta\, d\delta\, d\psi\,.$$

Remark. We choose $\mathcal{N}$ large enough in order to take the radius of the spherical cup $J_j(\eta)$ with $C_1 < \frac{10^{-5}}{2}$.

Estimate for the domain $\mathcal{A}_1$.

If $C_1 < \frac{10^{-5}}{2}$ then $|\cos\psi| \le 2C_1 < 10^{-5}$. We have

$$\begin{aligned} 1 - 10^{-9} > |\cos(\psi - \zeta)| &\ge |\sin\psi\,\sin\zeta\,| - |\cos\psi\,\cos\zeta\,| \\ &\ge \sqrt{1 - 10^{-10}}\,\sin\zeta\, - 10^{-5}|\cos\zeta\,| \end{aligned}$$

$$= \sqrt{1-10^{-10}}\,\sqrt{1-\cos^2\zeta}\, - 10^{-5}|\cos\zeta\,|\,,$$

and

$$\cos^2\zeta + 2\cdot 10^{-5}(1-10^{-9})|\cos\zeta\,| + (1-10^{-9})^2 - (1-10^{-10}) > 0\,,$$

where the discriminant of the corresponding quadratic polynomial in $|\cos\zeta\,|$ is

$$\Delta = 4\cdot 10^{-9}(1-10^{-10})(2-10^{-9}) > 0\,,$$

in such a way that ζ satisfies

$$|\cos\zeta\,| > -(1-10^{-9})10^{-5} + \sqrt{10^{-9}(1-10^{-10})(2-10^{-9})} > 3\cdot 10^{-5}\,.$$

Keeping in mind that $|\cos\psi| < 10^{-5}$, $|\cos\zeta\,| > 3\cdot 10^{-5}$, it is true that

$$|\sin\zeta\,\cos\psi\,\cos(\delta-\gamma) - \sin\psi\,\cos\zeta| \tag{3.50}$$

$$\geq |\sin\psi\,\cos\zeta| - |\sin\zeta\,\cos\psi\,\cos(\delta-\gamma)| \geq |\sin\psi\,\cos\zeta| - |\sin\zeta\,\cos\psi| \tag{3.51}$$

$$\geq 3\cdot 10^{-5}\sqrt{1-10^{-10}} - 10^{-5}\sqrt{1-9\cdot 10^{-10}} \sim 2\cdot 10^{-5}\,. \tag{3.52}$$

For each j, s, δ fixed we take the change of variables $(\zeta\,,\,\gamma\,,\,\psi) \,\to\, \lambda = (\lambda_1, \lambda_2, \lambda_3)$, given by

$$\lambda = \frac{h(s)}{2}\,((\sin\psi\,\cos\delta\,+\,\sin\zeta\,\cos\gamma)\,e_1 \tag{3.53}$$

$$+\,(\sin\psi\,\sin\delta\,+\,\sin\zeta\,\sin\gamma)\,e_2\,+\,(\cos\psi\,+\,\cos\zeta)\,e_3). \tag{3.54}$$

We have

$$\left|\frac{\partial\,(\lambda_1\,,\,\lambda_2\,,\,\lambda_3)}{\partial\,(\zeta\,,\,\gamma\,,\,\psi)}\right| = \frac{h(s)^3\sin\zeta\,\,|\sin\zeta\,\cos\psi\,\cos(\delta-\gamma) - \sin\psi\,\cos\zeta|}{8}\,.$$

By Fubini's and Toneli's theorems and estimates (3.50),(3.51),(3.52), we may write

$$\begin{aligned}
&\int\!\!\int\!\!\int\!\!\int_{\mathcal{A}_1} h(s)^4\sin\psi\sin\zeta\,|\hat{q}(A(j,s,\psi,\delta,\zeta,\gamma))|^2 d\gamma\,d\zeta\,d\delta\,d\psi\\
&\leq C\int_{-\varepsilon_0}^{\varepsilon_0}\int_{\mathbb{R}^3} h(s)|\hat{q}(\lambda)|^2 d\lambda\,d\delta\\
&= C\,h(s)\int_{\mathbb{R}^3}|\hat{q}(\lambda)|^2 d\lambda\,.
\end{aligned}$$

Estimate for the domain $\mathcal{A}_2$.

Now we apply the change of variables $(\psi\,,\,\delta\,,\,\gamma) \,\to\, \lambda = (\lambda_1, \lambda_2, \lambda_3)$, given by (3.53)-(3.54) for each j, s, ζ fixed. It holds

$$\left|\frac{\partial\,(\lambda_1\,,\,\lambda_2\,,\,\lambda_3)}{\partial\,(\psi\,,\,\delta\,,\,\gamma)}\right| = \frac{h(s)^3\sin^2\psi\,\sin\zeta\,\,|\sin(\delta-\gamma)|}{8}\,.$$

On one hand, since $|\psi - \frac{\pi}{2}| \leq \varepsilon_0$, we have that $|\cos\psi| \leq 2C_1$. Hence the sinus of ψ is lower bounded by a strictly positive constant.

On the other hand, since $\gamma \in X^*(\zeta, \psi, \delta)$ we know that the expression

$$\begin{aligned}
&\sin\psi\cos\delta\sin\zeta\cos\gamma + \sin\psi\sin\delta\sin\zeta\sin\gamma + \cos\psi\cos\zeta\\
&= \cos(\psi-\zeta)\cos(\delta-\gamma) + \cos\psi\,\cos\zeta\,(1-\cos(\delta-\gamma))
\end{aligned}$$

takes values between $-(1-1/5000)$ and $1-1/80000$. We have

$$\begin{aligned}1-\frac{1}{80000} &\geq |\cos(\psi-\zeta)\cos(\delta-\gamma)+\cos\psi\,\cos\zeta\,(1-\cos(\delta-\gamma))| \\ &\geq |\cos(\psi-\zeta)\cos(\delta-\gamma)|-|\cos\psi\,\cos\zeta|\,(1-\cos(\delta-\gamma)) \\ &\geq (1-10^{-9})|\cos(\delta-\gamma)|-2C_1\,|\cos\zeta|\,(1-\cos(\delta-\gamma)).\end{aligned}$$

Provided that $\cos(\delta-\gamma)\geq 0$ it holds

$$1-\frac{1}{80000}\geq (1-10^{-9}+2C_1|\cos\zeta|)\cos(\delta-\gamma)-2C_1|\cos\zeta|,$$

and hence,

$$\cos(\delta-\gamma)\leq\frac{1-1/80000+2C_1|\cos\zeta|}{1-10^{-9}+2C_1|\cos\zeta|}\leq\frac{1-1/80000+2C_1}{1-10^{-9}}.$$

Nevertheless, if $\cos(\delta-\gamma)<0$ we may write

$$\begin{aligned}1-\frac{1}{80000} &\geq -(1-10^{-9}-2C_1|\cos\zeta|)\cos(\delta-\gamma)-2C_1|\cos\zeta| \\ &\geq -(1-10^{-9}-2C_1)\cos(\delta-\gamma)-2C_1|\cos\zeta|.\end{aligned}$$

The choice $C_1<\frac{10^{-5}}{2}$ allows us to write

$$\cos(\delta-\gamma)\geq -\frac{1-1/80000+2C_1|\cos\zeta|}{1-10^{-9}-2C_1}\geq -\frac{1-1/80000+2C_1}{1-10^{-9}-2C_1},$$

and

$$|\cos(\delta-\gamma)|\leq\frac{1-1/80000+2C_1}{1-10^{-9}-2C_1}.$$

Hence $|\sin(\delta-\gamma)|$ is bounded below by a strictly positive constant which only depends on C_1.

By Fubini's and Toneli's theorems, we have

$$\begin{aligned}&\int\int\int\int_{\mathcal{A}_2} h(s)^4\sin\psi\sin\zeta\,|\hat{q}(A(j,s,\psi,\delta,\zeta,\gamma))|^2 d\gamma\,d\zeta\,d\delta\,d\psi \\ &\leq C\int_0^{\pi}\int_{\mathbb{R}^3} h(s)|\hat{q}(\lambda)|^2 d\lambda\,d\zeta \\ &= C\,h(s)\int_{\mathbb{R}^3}|\hat{q}(\lambda)|^2 d\lambda\,,\end{aligned}$$

and claim 3.1 follows.

□

We return to the expression (3.45)-(3.46). In (3.46) we write the variables ξ',ϕ' in spherical coordinates as we did in (3.48), (3.49):

$$\begin{aligned}\xi' &= \frac{\xi+sz}{2}+\frac{h(s)}{2}\left(\sin\Theta\cos\theta\,e_1+\sin\Theta\sin\theta\,e_2+\cos\Theta\,e_3\right), \\ \phi' &= \frac{\xi+sz}{2}+\frac{h(s)}{2}\left(\sin\zeta'\cos\gamma'\,e_1+\sin\zeta'\sin\gamma'\,e_2+\cos\zeta'\,e_3\right),\end{aligned}$$

where $\Theta,\zeta'\in[0,\pi]$, $\theta,\gamma'\in(-\pi,\pi]$. It holds

$$d\sigma(\xi')d\sigma(\phi')=h(s)^4\sin\Theta\sin\zeta'\,d\theta\,d\Theta\,d\gamma'\,d\zeta'.$$

For each $\Theta\in[0,\pi]$ and ζ',γ', we define

$$Y^*(\Theta,\zeta',\gamma'):=\{\theta\in(-\pi,\pi]\,:\,\xi'\in Y_{\xi+sz}(\phi')\}$$

$$= \big\{ \theta \in (-\pi, \pi] \,:\, \sin\Theta \cos\theta \sin\zeta' \cos\gamma' \tag{3.55}$$
$$+ \sin\Theta \sin\theta \sin\zeta' \sin\gamma' + \cos\Theta\cos\zeta' \geq -\big(1 - 1/5000\big) \big\}. \tag{3.56}$$

(3.45)-(3.46) is bounded by

$$\begin{aligned} & C\,\|q\|_{L^2}^2 \int_{\mathbb{S}^1} \int_0^\infty h(s)^{2\beta} \int_0^\pi \int_{-\pi}^\pi \int_0^\pi \int_{Y^*(\Theta,\zeta',\gamma')} \sin\Theta \sin\zeta' \\ & \times |\hat{q}(B(j,s,\Theta,\theta,\zeta',\gamma'))|^2 d\theta\, d\Theta\, d\gamma'\, d\zeta'\, s\, ds\, d\sigma(z) \\ & = C\,\|q\|_{L^2}^2 \int_0^\infty h(s)^{2\beta} \int_0^\pi \int_{-\pi}^\pi \int_0^\pi \int_{Y^*(\Theta,\zeta',\gamma')} \sin\Theta \sin\zeta' \\ & \quad \times |\hat{q}(B(j,s,\Theta,\theta,\zeta',\gamma'))|^2 d\theta\, d\Theta\, d\gamma'\, d\zeta'\, s\, ds\,, \end{aligned}$$

where

$$\begin{aligned} B(j,s,\Theta,\theta,\zeta',\gamma') := \frac{-h(s)}{2} & \,((\sin\Theta\, \cos\theta \,+\, \sin\zeta'\, \cos\gamma')\, e_1 \\ & + (\sin\Theta\, \sin\theta \,+\, \sin\zeta'\, \sin\gamma')\, e_2 \,+\, (\cos\Theta \,+\, \cos\zeta')\, e_3). \end{aligned}$$

Next for each j, ζ', γ' fixed we change variables $(s, \Theta, \theta) \to \mu = (\mu_1, \mu_2, \mu_3)$ given by

$$\mu = B(j,s,\Theta,\theta,\zeta',\gamma').$$

The Jacobean of this transformation is given by

$$\begin{aligned} \left| \frac{\partial(\mu_1,\mu_2,\mu_3)}{\partial(s,\Theta,\theta)} \right| = \frac{s\,h(s)\sin\Theta}{8} & \,(1 + \sin\Theta\, \sin\zeta'\, \cos\theta\, \cos\gamma' \\ & + \sin\Theta\, \sin\zeta'\, \sin\theta\, \sin\gamma' \,+\, \cos\Theta\, \cos\zeta')\,. \end{aligned}$$

Notice that the mentioned change involves an expression for s in terms of μ which depends on the parameters j, ζ', γ'. Hence, the function h has the same parametric dependence

$$h(s) = h(\mu, j, \zeta', \gamma').$$

Nevertheless, for $\theta \in Y^*(\Theta, \zeta', \gamma')$ it holds

$$h(s) \sim |\mu| = |B(j,s,\Theta,\theta,\zeta',\gamma')| \tag{3.57}$$

thanks to the condition stated in (3.55)-(3.56), where

$$\begin{aligned} |B(j,s,\Theta,\theta,\zeta',\gamma')| = \frac{|h(s)|}{2} & \,(1 + \sin\Theta\, \sin\zeta'\, \cos\theta\, \cos\gamma' \\ & + \sin\Theta\, \sin\zeta'\, \sin\theta\, \sin\gamma' \,+\, \cos\Theta\, \cos\zeta')^{\frac{1}{2}}. \end{aligned}$$

Indeed, property (3.57) is a reminiscence of the condition $\frac{|\eta|}{100} \leq |\eta - \phi' - \xi'| \leq |\eta|$ held for $\xi' \in Y_\eta(\phi')$, see (3.44).

We conclude that

$$\begin{aligned} G_j(\xi) & \leq C\,\|q\|_{L^2}^2 \int_0^\pi \int_{-\pi}^\pi \int_{\mathbb{R}^3} |\mu|^{2\beta-1} |\hat{q}(\mu)|^2 d\mu\, d\gamma'\, d\zeta' \\ & = C\,\|q\|_{L^2}^2 \int_{\mathbb{R}^3} |\mu|^{2\beta-1} |\hat{q}(\mu)|^2 d\mu\,, \end{aligned}$$

and then, we have proved the estimate

$$\|R_{J_j}(q)\|_{\dot{W}^{\beta,2}} \leq C \|q\|_{\dot{W}^{-\frac{1}{2},2}} \|q\|_{L^2}^2 \|q\|_{\dot{W}^{\beta-\frac{1}{2},2}}\,,$$

and since $\mathcal{N}$ is an universal constant we also have

$$\|Q_{I_2}(q)\|_{\dot W^{\beta,2}} \le C\|q\|_{\dot W^{-\frac{1}{2},2}}\|q\|_{L^2}^2\|q\|_{\dot W^{\beta-\frac{1}{2},2}}\,.$$

This ends the proof of claim (3.23c) and estimate (3.23).

□

Proof of estimate (3.25) **.** This case is inspired on the method used to control the piece $Q'_{II}(q)$ of the cubic term from the Neumann-Born series in the three-dimensional case in [28].

Let us start by decomposing the set $IV(\eta)$ as follows: $IV(\eta) \subset IV_<(\eta) \cup IV_>(\eta)$, where

$$IV_<(\eta) := \left\{(\xi,\tau,\phi) \in IV(\eta)\,:\,|\xi|,|\tau|,|\phi| \le \left(\frac{1}{50}+\frac{1}{\sqrt{2}}\right)|\eta|\right\}\,,$$
$$IV_>(\eta) := \left\{(\xi,\tau,\phi) \in IV(\eta)\,:\,|\eta-\xi|,|\eta-\tau|,|\eta-\phi| \le \left(\frac{1}{50}+\frac{1}{\sqrt{2}}\right)|\eta|\right\}\,.$$

In fact, if $|\xi| \le \frac{1}{\sqrt{2}}|\eta|$ hence

$$|\phi| \le |\phi-\xi|+|\xi| \le \left(\frac{1}{\sqrt{2}}+\frac{1}{100}\right)|\eta|\,,\quad |\tau| \le |\tau-\phi|+|\phi| \le \left(\frac{2}{100}+\frac{1}{\sqrt{2}}\right)|\eta|\,,$$

and if $|\xi| \ge \frac{1}{\sqrt{2}}|\eta|$ then $|\eta-\xi| \le \frac{1}{\sqrt{2}}|\eta|$, and it holds

$$|\eta-\phi| \le |\eta-\xi|+|\xi-\phi| \le \left(\frac{1}{\sqrt{2}}+\frac{1}{100}\right)|\eta|\,,\quad |\eta-\tau| \le |\eta-\phi|+|\phi-\tau| \le \left(\frac{2}{100}+\frac{1}{\sqrt{2}}\right)|\eta|\,.$$

Taking the changes of variables $\xi = \eta-\xi'$, $\tau=\eta-\tau'$, $\phi=\eta-\phi'$ in the integral

$$\int\!\!\int\!\!\int_{IV_>(\eta)} |\hat q(\xi')\hat q(\eta-\tau')\hat q(\tau'-\phi')\hat q(\phi'-\xi')|\,d\sigma(\xi')d\sigma(\tau')d\sigma(\phi')\,,$$

we notice that $\widehat{Q_{IV_>}(q)}(\eta) = \widehat{Q_{IV_<}(q)}(\eta)$, and then $\widehat{Q_{IV}(q)}(\eta) \le 2\,\widehat{Q_{IV_<}(q)}(\eta)$. We take another decomposition: $IV_<(\eta) \subset \bigcup_{k=1}^{+\infty}\left(IV_k(\eta)\cup\widetilde{IV}_k(\eta)\right)$, being

$$IV_k(\eta) := \left\{(\xi,\tau,\phi)\in IV_<(\eta)\,:\,|\phi|\le|\xi|\sim 2^{-k}|\eta|\right\}\,,$$
$$\widetilde{IV}_k(\eta) := \left\{(\xi,\tau,\phi)\in IV_<(\eta)\,:\,|\xi|\le|\phi|\sim 2^{-k}|\eta|\right\}\,.$$

It holds

$$\|Q_{IV}(q)\|_{\dot W^{\beta,2}} \le 2\sum_{k=1}^{+\infty}\left(\|Q_{IV_k}(q)\|_{\dot W^{\beta,2}}+\|Q_{\widetilde{IV}_k}(q)\|_{\dot W^{\beta,2}}\right)\,,$$

hence estimate (3.25) follows from the following claims:

$$\|Q_{IV_k}(q)\|_{\dot W^{\beta,2}} \le C\,2^{-\varepsilon k}\,\|q\|_{\dot W^{-\frac{1}{2},2}}\|q\|_{\dot W^{-\varepsilon,2}}\|q\|_{L^2}\|q\|_{\dot W^{\beta-\frac{1}{2}+\varepsilon,2}}\,, \tag{3.25a}$$
$$\|Q_{\widetilde{IV}_k}(q)\|_{\dot W^{\beta,2}} \le C\,2^{-\varepsilon k}\,\|q\|_{\dot W^{-\frac{1}{2},2}}\|q\|_{\dot W^{-\varepsilon,2}}\|q\|_{L^2}\|q\|_{\dot W^{\beta-\frac{1}{2}+\varepsilon,2}} \tag{3.25b}$$
$$+\,C\,2^{-\varepsilon k}\,\|q\|_{\dot W^{-\frac{1}{2}-\varepsilon,2}}\|q\|_{L^2}^2\|q\|_{\dot W^{\beta-\frac{1}{2}+\varepsilon,2}}\,, \tag{3.25c}$$

provided that $\varepsilon > 0$.

Proof of claim (3.25a) **.** By Cauchy-Schwartz inequality,

$$\widehat{Q_{IV_k}(q)}(\eta) \leq \frac{1}{|\eta|^3} \left(\int\int\int_{IV_k(\eta)} |\hat{q}(\xi)\hat{q}(\eta-\tau)|^2 d\sigma(\xi) d\sigma(\tau) d\sigma(\phi) \right)^{\frac{1}{2}}$$
$$\times \left(\int\int\int_{IV_k(\eta)} |\hat{q}(\tau'-\phi')\hat{q}(\phi'-\xi')|^2 d\sigma(\xi') d\sigma(\tau') d\sigma(\phi') \right)^{\frac{1}{2}},$$

where

$$IV_k(\eta) = \{(\xi,\tau,\phi) \in \Gamma(\eta)^3 : |\phi-\xi| \leq |\eta|/100,\, |\phi-\tau| \leq |\eta|/100,$$
$$|\xi|, |\tau|, |\phi| \leq (1/\sqrt{2} + 1/50)|\eta|,\, |\phi| \leq |\xi| \sim 2^{-k}|\eta|\}.$$

For each $\eta \in \mathbb{R}^3$, $\phi' \in \Gamma(\eta)$ fixed, using as above the maximal operator we have

$$\int_{\Gamma(\eta)} |\hat{q}(\tau'-\phi')|^2 d\sigma(\tau') \leq C\|q\|_{L^2}^2 .$$

Since $\sigma(\Gamma(\eta)) = \pi\,|\eta|^2$ we also get

$$\|Q_{IV_k}(q)\|^2_{\dot{W}^{\beta,2}} \leq C\|q\|^2_{L^2} \int_{\mathbb{R}^3} |\eta|^{2\beta-4} \int_{B_k(\eta)} |\hat{q}(\xi)|^2 \int_{\mathcal{B}_\xi(\eta)} |\hat{q}(\eta-\tau)|^2 d\sigma(\tau)\, d\sigma(\xi) \tag{3.58}$$
$$\times \int\int_{\mathcal{C}_k(\eta)} |\hat{q}(\phi'-\xi')|^2 d\sigma(\xi') d\sigma(\phi') d\eta \tag{3.59}$$
$$\leq C\|q\|^2_{\dot{W}^{-\frac{1}{2},2}} \|q\|^2_{L^2} \int_{\mathbb{R}^3} |\eta|^{2\beta-4} \int_{B_k(\eta)} |\hat{q}(\xi)|^2 \tag{3.60}$$
$$\times\, 2^{-k}|\eta|^2 \int_{\mathcal{B}_\xi(\eta)} |\hat{q}(\eta-\tau)|^2 d\sigma(\tau) d\sigma(\xi) d\eta \tag{3.61}$$
$$\leq C 2^{-2k\varepsilon} \|q\|^2_{\dot{W}^{-\frac{1}{2},2}} \|q\|^2_{L^2} \int_{\mathbb{R}^3} |\hat{q}(\xi)|^2 \int_{\Lambda^*(\xi)} |\eta|^{2\beta-2} \tag{3.62}$$
$$\times \frac{|\eta|^{2\varepsilon}}{|\xi|^{2\varepsilon}} \int_{\mathcal{B}_\xi(\eta)} |\hat{q}(\eta-\tau)|^2 d\sigma(\tau) d\sigma(\eta) d\xi \tag{3.63}$$
$$\leq C 2^{-2k\varepsilon} \|q\|^2_{\dot{W}^{-\frac{1}{2},2}} \|q\|^2_{\dot{W}^{-\varepsilon\,,\,2}} \|q\|^2_{L^2} \|q\|^2_{\dot{W}^{\beta-\frac{1}{2}+\varepsilon\,,\,2}} \tag{3.64}$$

where

$$B_k(\eta) := \left\{ \xi \in \Gamma(\eta) \,:\, |\xi| \sim 2^{-k}|\eta|\,,\, |\xi| \leq \left(\frac{1}{50} + \frac{1}{\sqrt{2}}\right)|\eta| \right\},$$
$$\mathcal{B}_\xi(\eta) := \left\{ \tau \in \Gamma(\eta) \,:\, |\xi-\tau| \leq \frac{|\eta|}{50}\,,\, |\tau| \leq \left(\frac{1}{50} + \frac{1}{\sqrt{2}}\right)|\eta| \right\}, \tag{3.65}$$
$$\mathcal{C}_k(\eta) := \left\{ (\xi',\phi') \in \Gamma(\eta)^2 \,:\, |\phi'-\xi'| \leq \frac{|\eta|}{100}\,,\, |\phi'| \leq |\xi'| \sim 2^{-k}|\eta| \right\},$$
$$\Lambda^*(\xi) := \left\{ \eta \in \Lambda(\xi) \,:\, |\xi| \leq \left(\frac{1}{50} + \frac{1}{\sqrt{2}}\right)|\eta| \right\}. \tag{3.66}$$

The estimate (3.34) allows us to estimate (3.58)-(3.59) by (3.60)-(3.61). The step from (3.60)-(3.61) to (3.62)-(3.63) follows from the property $|\xi| \sim 2^{-k}|\eta|$ and the change of the order of integration in ξ and η through Lemma 6.1. Finally, we bound (3.62)-(3.63) by (3.64) applying part (ii) of Lemma 6.3.

□

Proof of claim (3.25b)-(3.25c)**.**

We split $\widetilde{IV}_k(\eta) = \widetilde{IV}_{k,a}(\eta) \cup \widetilde{IV}_{k,b}(\eta)$, where

$$\begin{aligned}\widetilde{IV}_{k,a}(\eta) &:= \Big\{(\xi,\tau,\phi) \in \widetilde{IV}_k(\eta) \,:\, |\xi| \geq 2^{-k-2}|\eta|\Big\} \\ &= \{(\xi,\tau,\phi) \in \Gamma(\eta)^3 : |\phi-\xi| \leq |\eta|/100,\, |\phi-\tau| \leq |\eta|/100, \\ &\qquad |\xi|,|\tau|,|\phi| \leq (1/\sqrt{2}\,+\,1/50)|\eta|,\, 2^{-k-2}|\eta| \leq |\xi| \leq |\phi| \sim 2^{-k}|\eta|\}\,, \\ \widetilde{IV}_{k,b}(\eta) &:= \Big\{(\xi,\tau,\phi) \in \widetilde{IV}_k(\eta) \,:\, |\xi| \leq 2^{-k-2}|\eta|\Big\} \\ &= \{(\xi,\tau,\phi) \in \Gamma(\eta)^3 : |\phi-\xi| \leq |\eta|/100,\, |\phi-\tau| \leq |\eta|/100, \\ &\qquad |\xi|,|\tau|,|\phi| \leq (1/\sqrt{2}\,+\,1/50)|\eta|,\, |\xi| \leq |\phi| \sim 2^{-k}|\eta|,\, |\xi| \leq 2^{-k-2}|\eta|\}.\end{aligned}$$

On the domain $\widetilde{IV}_{k,a}(\eta)$, $2^{-k-2}|\eta| \leq |\xi| \leq |\phi| \leq 2^{-k+1}|\eta|$ holds; hence following the steps of the proof for the domain $IV_k(\eta)$ we arrive at

$$\|Q_{\widetilde{IV}_{k,a}}(q)\|_{\dot W^{\beta,2}} \leq C\,2^{-\varepsilon k}\,\|q\|_{\dot W^{-\frac12,2}}\|q\|_{\dot W^{-\varepsilon\,,2}}\|q\|_{L^2}\|q\|_{\dot W^{\beta-\frac12+\varepsilon\,,2}}\,. \tag{3.67}$$

On the domain $\widetilde{IV}_{k,b}(\eta)$ we have $|\xi-\phi| \geq 2^{-k-2}|\eta|$, in fact

$$|\xi-\phi| \geq |\phi| - |\xi| \geq 2^{-k-1}|\eta| - 2^{-k-2}|\eta| = 2^{-k-2}|\eta|\,.$$

In this case, we can bound $\|Q_{\widetilde{IV}_{k,b}}(q)\|^2_{\dot W^{\beta,2}}$ by a similar expression to (3.58)-(3.59) replacing $B_k(\eta)$ by the set $\{\xi \in \Gamma(\eta) : |\xi| \leq 2^{-k-2}|\eta|,\, |\xi| \leq (\frac{1}{50}+\frac{1}{\sqrt2})|\eta|\}$ and the domain $\mathcal{C}_k(\eta)$ by the set

$$\left\{(\xi',\phi') \in \Gamma(\eta)^2 \,:\, 2^{-k-2}|\eta| \leq |\phi'-\xi'| \leq \frac{|\eta|}{100}\,,\, |\xi'| \leq |\phi'| \sim 2^{-k}|\eta|\right\}.$$

By the estimate (3.35), changing the order of integration in ξ and η by Lemma 6.1, multiplying and dividing by $\frac{|\eta|^{2\varepsilon}}{|\xi|^{2\varepsilon}}$ and applying that $\frac{|\xi|^{2\varepsilon}}{|\eta|^{2\varepsilon}} \leq 2^{2\varepsilon(-k-2)}$, and finally by part (ii) of Lemma 6.3 we obtain

$$\|Q_{\widetilde{IV}_{k,b}}(q)\|_{\dot W^{\beta,2}} \leq C\,2^{-\varepsilon k}\,\|q\|_{\dot W^{-\frac12-\varepsilon\,,2}}\|q\|^2_{L^2}\|q\|_{\dot W^{\beta-\frac12+\varepsilon\,,2}}\,. \tag{3.68}$$

The expressions (3.67) and (3.68) lead up to claim (3.25b)-(3.25c).

This ends the proof of estimate (3.25) and Proposition 3.

□

4. Proof of Theorem 5 (remainder term $\alpha < n/2$).

The control of the remainder term $\mathbf{R}_l$, where l is as in the statement of Theorem 5, follows from the next proposition by choosing C_0 large enough in (1.7). For $C_0 > (2\|q\|_{W^{\alpha,2}})^4$ we obtain the convergence of the series

$$\sum_{j=l}^{+\infty}\left(2C_0^{-\frac14}\,\|q\|_{W^{\alpha,2}}\right)^j\,.$$

Proposition 4. *Let $n \in \{2,3\}$, $q \in W^{\alpha,2}(\mathbb{R}^n)$ compactly supported and $0 \leq \alpha < n/2$. Assume that $C_0 > 1$, $j \geq 4$ if $n=2$ and $j \geq 5$ if $n=3$. Then, for any $\beta \in \mathbb{R}$ such that $\beta < \alpha+1$, it holds:*

$$\|\widetilde{Q}_j(q)\|_{\dot W^{\beta,2}} \leq C(\alpha,\beta)\,C_0^{17/4}(2C_0^{-\frac14}\|q\|_{W^{\alpha,2}})^j\,\|q\|_{L^2}\|q\|^{-1}_{W^{\alpha,2}}. \tag{4.1}$$

Proof of Proposition 4. We write $R_{\theta,k}f(x) = e^{-ik\theta\cdot x}R_k(e^{ik\theta\cdot(\cdot)}f(\cdot))(x)$. It holds

$$\widehat{Q_j(q)}(\xi) = \int_{\mathbb{R}^n} e^{ik\theta\cdot y}(qR_k)^{j-1}(q(\cdot)e^{ik\theta\cdot(\cdot)})(y)dy = \int_{\mathbb{R}^n} e^{2ik\theta\cdot y}(qR_{\theta,k})^{j-1}(q)(y)\,dy\,,$$

where $k = |\xi|/2$ and $\theta = -\xi/|\xi|$. Let $\gamma \in \mathbb{R}$ be such that $\gamma < \beta_j$, being

$$\beta_j := \begin{cases} \frac{3}{4}(j-2) + \frac{\alpha}{4}(j-1)\,, & \text{if } \alpha \le \frac{1}{2} \text{ and } n = 2\,, \\ (j-3)(\frac{3}{4} + \frac{\alpha}{4}) + 1\,, & \text{if } \frac{1}{2} \le \alpha \le 1 \text{ and } n = 2\,, \\ \frac{j-2}{2} + (j-1)\frac{\alpha}{3} - \frac{1}{2}\,, & \text{if } 0 \le \alpha \le \frac{3}{4} \text{ and } n = 3\,, \\ (j-3)\left(\frac{1}{2} + \frac{\alpha}{3}\right) + \frac{1}{2}\,, & \text{if } \frac{3}{4} \le \alpha \le \frac{3}{2} \text{ and } n = 3\,. \end{cases}$$

Taking the change of variables $\xi = -2k\theta$ with $k \ge 0\,,\, \theta \in S^{n-1}$, we have

$$\begin{aligned} \|\widetilde{Q}_j(q)\|^2_{\dot{W}^{\gamma,2}} &= \int_{\mathbb{R}^n} |\xi|^{2\gamma}|\widehat{\widetilde{Q}_j(q)}(\xi)|^2 d\xi = \int_{\mathbb{R}^n} |\xi|^{2\gamma}\chi^*(\xi)|\widehat{Q_j(q)}(\xi)|^2 d\xi \\ &\le C_n\, 2^{2\gamma}\int_{k=\frac{C_0}{2}}^{+\infty} k^{2\gamma+n-1}\int_{S^{n-1}} \|(qR_{\theta,k})^{j-1}(q)\|^2_{L^1}d\sigma(\theta)dk\,. \end{aligned} \tag{4.2}$$

By Cauchy-Schwartz inequality, $\|(qR_{\theta,k})^{j-1}(q)\|_{L^1} \le C\|q\|_{L^2}\|R_{\theta,k}(qR_{\theta,k})^{j-2}(q)\|_{L^2}$. Using the estimate given by Lemma 3.4 in [26] for the operator $R_{\theta,k}$ and the next inequality for products of Sobolev spaces due to Zolesio

$$\|fg\|_{W^{\alpha_3,p}} \le C\,\|f\|_{W^{\alpha_1,p_1}}\|g\|_{W^{\alpha_2,p_2}}\,,$$

where $\alpha_1, \alpha_2, \alpha_3 \ge 0$, $\alpha_3 \le \alpha_j$, $p_j > p$, $j = 1, 2$, $\alpha_1 + \alpha_2 - \alpha_3 \ge n(1/p_1 + 1/p_2 - 1/p) \ge 0$, we arrive at

$$\|R_{\theta,k}(qR_{\theta,k})^{j-2}(q)\|_{L^2} \le Ck^{\gamma_j}\|q\|^{j-1}_{W^{\alpha,2}}\,,$$

where $\gamma_j := -(j-1) + \frac{n-1}{2}(j-3)(1/2 - \alpha/n) + \frac{n-1}{2}\max\{0,\, \frac{1}{2} - \frac{2\alpha}{n}\}$. All this leads us up to

$$\|\widetilde{Q}_j(q)\|^2_{\dot{W}^{\gamma,2}} \le C\,2^{2\gamma}\int_{k=\frac{C_0}{2}}^{+\infty} k^{2\gamma+n-1+2\gamma_j}\,dk\,\|q\|^2_{L^2}\|q\|^{2j-2}_{W^{\alpha,2}}\,,$$

where the integral converges if $2\gamma + 2\gamma_j + n < 0$, that is to say, if $\gamma < \beta_j$. Notice that $\beta_j = -\frac{n}{2} - \gamma_j$. In this way, we have proved that, for $\gamma < \beta_j$, it holds

$$\|\widetilde{Q}_j(q)\|_{\dot{W}^{\gamma,2}} \le C\frac{2^\gamma}{\sqrt{\beta_j - \gamma}}\left(\frac{C_0}{2}\right)^{\gamma-\beta_j}\|q\|_{L^2}\|q\|^{j-1}_{W^{\alpha,2}}\,. \tag{4.3}$$

Let $\varepsilon = \varepsilon(\alpha,\beta) := (\alpha+1) - \beta > 0$. Keeping in mind $2\beta = 2(\beta_j - \varepsilon) + 2(\alpha + 1 - \beta_j)$ and $\alpha + 1 \le \beta_j$ for our j, we write

$$\begin{aligned} \|\widetilde{Q}_j(q)\|_{\dot{W}^{\beta,2}} &\le C_0^{\alpha+1-\beta_j}\|\widetilde{Q}_j(q)\|_{\dot{W}^{\beta_j-\varepsilon,2}} \\ &\le C\,C_0^{\alpha+1-\beta_j}\frac{2^{\beta_j-\varepsilon}}{\sqrt{\varepsilon}}\left(\frac{C_0}{2}\right)^{-\varepsilon}\|q\|_{L^2}\|q\|^{j-1}_{W^{\alpha,2}} \\ &= C(\alpha,\beta)\,2^{\beta_j}\,C_0^{\beta-\beta_j}\|q\|_{L^2}\|q\|^{j-1}_{W^{\alpha,2}}\,, \end{aligned}$$

where the last inequality follows from formula (4.3) in the case $\gamma = \beta_j - \varepsilon$. In our setting, $2^{\beta_j} \le 2^j$. Moreover, $\beta - \beta_j < \alpha + 1 - \beta_j \le -\frac{1}{4}j + \frac{17}{4}$, for our j, and $C_0 > 1$. We have proved (4.1).

$\square$

5. The case $\alpha \geq n/2$.

In this section we are going to extend Theorem 4, Theorem 5 and estimates (2.5), (2.6), (2.7) and (3.1)-(3.2) for any $\alpha \geq 0$. Then Theorem 1 will follow from these estimates for any $\alpha \geq 0$. We start with a Leibniz' type formula for derivatives of $Q_j(q)$ which we state as follows

Theorem 6. *Assume that $\alpha \in \mathbb{N}^n$, $j \in \mathbb{Z}$, $j \geq 2$ and let $q \in W^{|\alpha|,2}(\mathbb{R}^n)$ be a compactly supported function. Then*

$$D^\alpha Q_j(q) = \sum_{\substack{\beta_1 + \ldots + \beta_j = \alpha \\ \beta_1, \ldots, \beta_j \geq 0}} \frac{\alpha!}{\beta_1! \cdot \ldots \cdot \beta_j!} Q_j(D^{\beta_1} q, \ldots, D^{\beta_j} q).$$

Remark. From the proof of Theorem 6 one also deduces the formula

$$D^\alpha \widetilde{Q}_j(q) = \sum_{\substack{\beta_1 + \ldots + \beta_j = \alpha \\ \beta_1, \ldots \beta_j \geq 0}} \frac{\alpha!}{\beta_1! \cdot \ldots \cdot \beta_j!} \widetilde{Q}_j(D^{\beta_1} q, \ldots, D^{\beta_j} q), \tag{5.1}$$

for the same hypotheses on q.

Proof of Theorem 6. Writing the resolvent R_k as the convolution operator with the outgoing fundamental solution to the Helmholtz equation (see [6], [27])

$$\phi_k(x) = C_n k^{(n-2)/2} \frac{H^{(1)}_{(n-2)/2}(k|x|)}{|x|^{(n-2)/2}}, \tag{5.2}$$

where $C_n = \frac{1}{2i(2\pi)^{(n-2)/2}}$ and $H^{(1)}_{(n-2)/2}$ denotes the Hankel function of the first kind and order $(n-2)/2$ (see [35]), we have

$$\begin{aligned} \widehat{Q_j(q)}(-2k\theta) &= \int_{\mathbb{R}^n} e^{ik\theta\cdot y} q(y) (R_k q)^{j-1} (e^{ik\theta\cdot(\cdot)})(y)\, dy \\ &= \int_{(\mathbb{R}^n)^j} e^{ik\theta\cdot x_1} q(x_1) \prod_{l=1}^{j-1} \left(\phi_k(x_l - x_{l+1}) q(x_{l+1})\right) e^{ik\theta\cdot x_j}\, dx, \end{aligned}$$

and

$$\mathcal{F}\left(Q_j(f_1, \ldots, f_j)\right)(-2k\theta) \tag{5.3}$$

$$= \int_{(\mathbb{R}^n)^j} e^{ik\theta\cdot x_1} f_1(x_1) \prod_{l=1}^{j-1} \left(\phi_k(x_l - x_{l+1}) f_{l+1}(x_{l+1})\right) e^{ik\theta\cdot x_j}\, dx, \tag{5.4}$$

where $dx = dx_1 \cdot \ldots \cdot dx_j$ and $x_l \in \mathbb{R}^n$, for any $l = 1, ..., j$. We know that $\mathcal{F}\left(D^\alpha Q_j(q)\right)(-2k\theta) = (-i2k\theta)^\alpha \widehat{Q_j(q)}(-2k\theta)$. Taking the change $x_l = x_1 + y_l$, $2 \leq l \leq j$, it holds

$$\begin{aligned} &\widehat{Q_j(q)}(-2k\theta) \\ &= \int_{(\mathbb{R}^n)^j} e^{i2k\theta\cdot x_1} q(x_1) \prod_{l=1}^{j-1} \left(\phi_k(x_l - x_{l+1}) q(x_{l+1}) e^{-ik\theta\cdot(x_l - x_{l+1})}\right) dx \\ &= \int_{\mathbb{R}^n} \int_{(\mathbb{R}^n)^{j-1}} e^{i2k\theta\cdot x_1} q(x_1) \prod_{l=1}^{j-1} \left(\phi_k(y_l - y_{l+1}) q(x_1 + y_{l+1}) e^{-ik\theta\cdot(y_l - y_{l+1})}\right) dy dx_1, \end{aligned}$$

where $y_1 = 0$ and $dy = dy_j \cdot \ldots \cdot dy_2$. Integrating by parts, we have

$$(-i2k\theta)^\alpha \, \widehat{Q_j(q)}(-2k\theta)$$

$$= (-1)^{|\alpha|} \int_{\mathbb{R}^n} \int_{(\mathbb{R}^n)^{j-1}} (D^\alpha_{x_1} e^{i2k\theta\cdot x_1})\, q(x_1) \prod_{l=1}^{j-1} \Big(\phi_k(y_l - y_{l+1}) q(x_1 + y_{l+1}) e^{-ik\theta\cdot (y_l - y_{l+1})} \Big)\, dydx_1$$

$$= \int_{\mathbb{R}^n} \int_{(\mathbb{R}^n)^{j-1}} e^{i2k\theta\cdot x_1}\, D^\alpha_{x_1} \left[q(x_1) \prod_{l=1}^{j-1} \Big(\phi_k(y_l - y_{l+1}) q(x_1 + y_{l+1}) e^{-ik\theta\cdot (y_l - y_{l+1})} \Big) \right] dydx_1$$

$$= \sum_{\substack{\beta_1 + \ldots + \beta_j = \alpha \\ \beta_1, \ldots, \beta_j \geq 0}} \frac{\alpha!}{\beta_1! \cdot \ldots \cdot \beta_j!} \int_{\mathbb{R}^n} \int_{(\mathbb{R}^n)^{j-1}} e^{i2k\theta\cdot x_1}\, D^{\beta_1} q(x_1)$$

$$\times \prod_{l=1}^{j-1} \Big(\phi_k(y_l - y_{l+1}) e^{-ik\theta\cdot (y_l - y_{l+1})} D^{\beta_{l+1}} q(x_1 + y_{l+1}) \Big)\, dydx_1\,,$$

where we have applied Leibniz' formula:

$$D^\alpha (f_1 \cdot \ldots \cdot f_k) = \sum_{\substack{\beta_1 + \cdots + \beta_k = \alpha \\ \beta_1, \ldots, \beta_k \geq 0}} \frac{\alpha!}{\beta_1! \cdot \ldots \cdot \beta_k!}\, D^{\beta_1} f_1 \cdot \ldots \cdot D^{\beta_k} f_k\,.$$

Finally,

$$(-i2k\theta)^\alpha\, \widehat{Q_j(q)}(-2k\theta) = \sum_{\substack{\beta_1 + \ldots + \beta_j = \alpha \\ \beta_1, \ldots, \beta_j \geq 0}} \frac{\alpha!}{\beta_1! \cdot \ldots \cdot \beta_j!} \int_{(\mathbb{R}^n)^j} e^{ik\theta\cdot x_1} D^{\beta_1} q(x_1)$$

$$\times \prod_{l=1}^{j-1} \big(\phi_k(x_l - x_{l+1}) D^{\beta_{l+1}} q(x_{l+1}) \big)\, e^{ik\theta\cdot x_j}\, dx\,,$$

and remembering the expression (5.3)-(5.4), we end the proof of Theorem 6.

□

To prove Theorem 1 in case $\alpha \geq n/2$ we use induction on $[\alpha]$. We need to use the boundedness of the j-multiple scattering operators, see the notation, when acting on $(q_1, ... q_j)$ where $j-1$ of them are equal to q. Namely

Proposition 5. *Let $n \in \{2,3\}$, $\alpha \in \mathbb{R}$ with $0 \leq \alpha < n/2$ and let us suppose that $q_1, q_2 \in W^{\alpha\,,2}(\mathbb{R}^n)$ are compactly supported functions. Then $Q_2(q_1, q_2) \in W^{\beta\,,2}(\mathbb{R}^n) + C^\infty(\mathbb{R}^n)$, for any $\beta \in \mathbb{R}$ such that $0 \leq \beta < \alpha + \frac{1}{2}$. Moreover, there exists a constant $C(\alpha, \beta, q_1, q_2) > 0$ which just depends of α, β and the supports of q_1, q_2 such that*

$$\|\widetilde{Q}_2(q_1, q_2)\|_{\dot{W}^{\beta,2}} \leq C(\alpha, \beta, q_1, q_2)\, \max\{\|q_1\|^2_{W^{\alpha,2}}, \|q_2\|^2_{W^{\alpha,2}}\}\,.$$

Proposition 5 follows by polarization of estimate (2.5).

Next propositions 6, 7 and 8 follow from the proofs of the analogous estimates (2.6), (2.7), (3.1)-(3.2) and Proposition 4.

Proposition 6. *Let $n \in \{2,3\}$, $\alpha \in \mathbb{R}$ with $0 \leq \alpha < n/2$ and q_1, q_2, q_3 as q_1 from Proposition 5. Then $Q_3(q_1, q_2, q_3) \in W^{\beta\,,2}(\mathbb{R}^n) + C^\infty(\mathbb{R}^n)$, for any $\beta \in \mathbb{R}$ holding $0 \leq \beta < \alpha + 1$ if $n = 2$ and $0 \leq \beta < \alpha + 1/2$ if $n = 3$. Moreover, there exists a constant $C(\alpha, \beta, q_1, q_2, q_3)$ that just depends of α, β and $\operatorname{supp} q_1, \operatorname{supp} q_2, \operatorname{supp} q_3$ such that*

$$\|\widetilde{Q}_3(q_1, q_2, q_3)\|_{\dot{W}^{\beta,2}} \tag{5.5}$$

$$\leq C(\alpha,\beta,q_1,q_2,q_3)\,\Big(\sum_{\sigma\in S_3}\|q_{\sigma(1)}\|_{L^2}\|q_{\sigma(2)}\|_{L^2}\|q_{\sigma(3)}\|_{\dot W^{\alpha,2}} \tag{5.6}$$

$$+\sum_{\tau\in S_3}\|q_{\tau(1)}\|_{\dot W^{-\frac12,2}}\|q_{\tau(2)}\|_{\dot W^{-\varepsilon,2}}\|q_{\tau(3)}\|_{\dot W^{\alpha,2}} \tag{5.7}$$

$$+\sum_{\omega\in S_3}\|q_{\omega(1)}\|_{\dot W^{-\frac12-\varepsilon,2}}\|q_{\omega(2)}\|_{L^2}\|q_{\omega(3)}\|_{\dot W^{\alpha,2}}\Big)\,, \tag{5.8}$$

where $\varepsilon=\alpha+1-\beta>0$ if $n=2$ and $\varepsilon=\alpha+\frac12-\beta>0$ if $n=3$.

Proposition 7. *Let $\alpha\in\mathbb{R}$ with $0\leq\alpha<3/2$ and q_1,q_2,q_3,q_4 as q_1 from Proposition 5 for $n=3$. Then $Q_4(q_1,q_2,q_3,q_4)\in W^{\beta\,,2}(\mathbb{R}^3)+C^\infty(\mathbb{R}^3)$, for any $\beta\in\mathbb{R}$ with $0\leq\beta<\alpha+1/2$. Moreover, there exists a constant $C(\alpha,\beta,q_1,q_2,q_3,q_4)>0$ just depending of α,β and the supports of q_1,q_2,q_3,q_4 such that*

$$\|\widetilde{Q}_4(q_1,q_2,q_3,q_4)\|_{\dot W^{\beta,2}}$$

$$\leq C(\alpha,\beta,q_1,q_2,q_3,q_4)\,\Big(\sum_{\sigma\in S_4}\|q_{\sigma(1)}\|_{L^2}\|q_{\sigma(2)}\|_{L^2}\|q_{\sigma(3)}\|_{L^2}\|q_{\sigma(4)}\|_{\dot W^{\alpha,2}} \tag{5.9}$$

$$+\sum_{\tau\in S_4}\|q_{\tau(1)}\|_{\dot W^{-\frac12,2}}\|q_{\tau(2)}\|_{L^2}\|q_{\tau(3)}\|_{L^2}\|q_{\tau(4)}\|_{\dot W^{\alpha,2}} \tag{5.10}$$

$$+\sum_{\omega\in S_4}\|q_{\omega(1)}\|_{\dot W^{-\frac12,2}}\|q_{\omega(2)}\|_{\dot W^{-\varepsilon,2}}\|q_{\omega(3)}\|_{L^2}\|q_{\omega(4)}\|_{\dot W^{\alpha,2}} \tag{5.11}$$

$$+\sum_{\rho\in S_4}\|q_{\rho(1)}\|_{\dot W^{-\frac12-\varepsilon,2}}\|q_{\rho(2)}\|_{L^2}\|q_{\rho(3)}\|_{L^2}\|q_{\rho(4)}\|_{\dot W^{\alpha,2}}\Big)\,, \tag{5.12}$$

where $\varepsilon=\alpha+\frac12-\beta>0$.

Proposition 8. *Let us assume that $n\in\{2,3\}$, $\alpha\in\mathbb{R}$, $0\leq\alpha<n/2$, $q_1,...,q_j\in W^{\alpha\,,2}(\mathbb{R}^n)$ are compactly supported functions and $C_0>1$. Hence, for any $\beta\in\mathbb{R}$ such that $\beta<\alpha+1$:*

$$\|\widetilde{Q}_j(q_1,...,q_j)\|_{\dot W^{\beta,2}}\leq C(\alpha,\beta)\,C_0^{17/4}\left(2C_0^{-\frac14}\max_{1\leq l\leq j}\|q_l\|_{W^{\alpha,2}}\right)^j, \tag{5.13}$$

where $j\geq 4$ if $n=2$ and $j\geq 5$ if $n=3$.

6. Appendix.

In this section we state two results, Lemma 6.1 and 6.2, which are often used in this work and state and prove an important result, Lemma 6.3, in order to demonstrate Proposition 3.

Let V be the submanifold of $\mathbb{R}^{2n}$ $V:=\{(\eta,\xi)\in\mathbb{R}^n\times\mathbb{R}^n:\xi\cdot(\xi-\eta)=0\}$. Then V can be viewed as a bundle of spherical sections $V=\{(\eta,\xi)\in\mathbb{R}^n\times\mathbb{R}^n:\eta\in\mathbb{R}^n,\xi\in\Gamma(\eta)\}$, or as a bundle of plane sections: $V=\{(\eta,\xi)\in\mathbb{R}^n\times\mathbb{R}^n:\xi\in\mathbb{R}^n,\eta\in\Lambda(\xi)\}$, where $\Gamma(\eta)$ and $\Lambda(\xi)$ are defined in (1.4) and (1.5). In this context, the following lemma from [28] allows us to change the order of integration in ξ and η.

Lemma 6.1. *Let $V=\{(\eta,\xi)\in\mathbb{R}^n\times\mathbb{R}^n:\xi\cdot(\xi-\eta)=0\}$. Let $d\sigma_\eta(\xi)$ be the measure on $\Gamma(\eta)$ induced by the n-dimensional Lebesgue measure $d\xi$ and let $d\sigma_\xi(\eta)$ be the measure on $\Lambda(\xi)$ induced by the n-dimensional Lebesgue measure $d\eta$. Then*

$$d\sigma_\eta(\xi)d\eta=\frac{|\eta|}{|\xi|}\,d\sigma_\xi(\eta)d\xi.$$

The following lemma in [28] is used several times in this work.

Lemma 6.2. *Assume that the support of q is contained in the unit ball. Then we have:*

(1) *If $\xi,\ \xi' \in \mathbb{R}^n$ satisfy $|\xi-\xi'|\le 3$, then $|\hat{q}(\xi)| \le CM\hat{q}(\xi')$.*

(2) $\|\hat{q}\|_{L^\infty} \le C\|\hat{q}\|_{L^2}$.

(3) *For $0<\gamma<\frac{n}{2}$, $\|q\|_{\dot{W}^{-\gamma,2}} \le C\|q\|_{L^2}$, where C depends on the size of the support of q.*

The following lemma is fundamental to control the spherical term $Q(q)$ of the quartic term.

Lemma 6.3. *Let $\xi\in\mathbb{R}^3\setminus\{0\}$, $\beta\in\mathbb{R}$, $\varepsilon>0$, $k\in\mathbb{N}$. We denote*

$$F_k(\xi) := \int_{\Lambda(\xi)} |\eta|^{2\beta-4} \int\int_{A_k(\eta)} |\hat{q}(\eta-\phi'-\xi')|^2 d\sigma(\xi')d\sigma(\phi')d\sigma(\eta)\,, \tag{6.1}$$

$$G(\xi) := \int_{\Lambda^*(\xi)} |\eta|^{2\beta-2+2\varepsilon} \int_{\mathcal{B}_\xi(\eta)} |\hat{q}(\eta-\tau)|^2 d\sigma(\tau)d\sigma(\eta)\,, \tag{6.2}$$

where $A_k(\eta)$, $\mathcal{B}_\xi(\eta)$, $\Lambda^*(\xi)$ are defined in (3.30), (3.65), (3.66) *respectively. Then*

(*i*) $F_k(\xi) \le C\,2^{-2k}\int_{\mathbb{R}^3}|\lambda|^{2\beta-1}|\hat{q}(\lambda)|^2d\lambda$, *for some constant C independent of ξ, k and q.*

(*ii*) $G(\xi) \le C\int_{\mathbb{R}^3}|\lambda|^{2\beta-1+2\varepsilon}|\hat{q}(\lambda)|^2d\lambda$, *for some constant C independent of ξ and q.*

Proof of Lemma 6.3.

• Proof of (*i*). For $\eta\in\Lambda(\xi)$ we write $\eta=\xi+sz$ and $h(s):=|\eta|=\left(|\xi|^2+s^2\right)^{\frac{1}{2}}$, where $s\ge 0$ and $z\in\{\xi\}^\perp$, $|z|=1$. For simplicity, we don't specify the dependence of variables with respect to ξ since it is fixed along the proof . It holds $d\sigma(\eta)=s\,dsd\sigma(z)$, where $d\sigma(z)$ denotes the measure on the unitary circumference $\mathbb{S}^1$ in the plane $\{\xi\}^\perp$. We have

$$F_k(\xi) = \int_0^\infty\int_{\mathbb{S}^1} h(s)^{2\beta-4}\int\int_{A_k(s,z)} |\hat{q}(\xi+sz-\phi'-\xi')|^2 d\sigma(\xi')d\sigma(\phi')d\sigma(z)s\,ds\,. \tag{6.3}$$

Fixing z,s we parametrize $\xi',\phi'\in\Gamma(\xi+sz)$ by $v,u\in\mathbb{S}^2$, respectively:

$$\xi' = \frac{\xi+sz}{2}+\frac{h(s)}{2}v\,,\ \phi' = \frac{\xi+sz}{2}+\frac{h(s)}{2}u\,,\quad u,v\in\mathbb{S}^2\,,$$

where $d\sigma(\xi')=C\,h(s)^2d\sigma(v)$, $d\sigma(\phi')=C\,h(s)^2d\sigma(u)$. The domain of integration for v,u is given by

$$\{(v,u)\in\mathbb{S}^2\times\mathbb{S}^2\,:\,|u-v|\le 2^{-k+2}\,,\,1+u\cdot v\ge 1/5000\}\,,$$

since $|\xi'-\phi'|\le 2^{-k+1}h(s)$ implies $|u-v|\le 2^{-k+2}$ and $|\xi+sz-\phi'-\xi'|\ge\frac{h(s)}{100}$ implies $1+u\cdot v\ge\frac{1}{5000}$. Let $\{D_{j,k}\,:\,j\in\{1,\dots,N_0 2^{2k}\}\}$ be a finite overlapping cover of the sphere $\mathbb{S}^2$ with overlapping constant independent of k such that $D_{j,k}$ is an spherical cup of diameter $\frac{2^{-k}}{50}$ and N_0 an appropriate constant. For each j we define

$$\widetilde{D}_{j,k} := \left\{u\in\mathbb{S}^2 : |u-v|\le 2^{-k+2}\,,\,1+u\cdot v\ge\frac{1}{5000}\,,\text{ for some } v\in D_{j,k}\right\}\,.$$

The expression (6.3) is bounded by

$$C\sum_{j=1}^{N_0 2^{2k}}\int_{\mathbb{S}^1}\int_{D_{j,k}}\int_0^{+\infty}\int_{\widetilde{D}_{j,k}} h(s)^{2\beta}\left|\hat{q}(-\frac{h(s)}{2}\ (u+v))\right|^2 d\sigma(u)\,s\,ds\,d\sigma(v)d\sigma(z)\,. \tag{6.4}$$

Notice that for every $j \in \{1, \ldots, N_0 2^{2k}\}$, $u \in \widetilde{D}_{j,k}$ and $v \in D_{j,k}$, we have $|u+v| \geq \frac{1}{100}$. In fact, since $u \in \widetilde{D}_{j,k}$ there exists a $v' \in D_{j,k}$ such that $1 + u \cdot v' \geq 1/5000$, and hence $|u+v'| \geq 1/50$. We have

$$\begin{aligned} |u+v| &\geq |u+v'| - |v-v'| \geq \frac{1}{50} - \operatorname{diam} D_{j,k} \\ &= \frac{1}{50} - \frac{2^{-k}}{50} \geq \frac{1}{50} - \frac{1}{100} = \frac{1}{100}. \end{aligned}$$

We take spherical coordinates for u with respect to the canonical reference of $\mathbb{R}^3$,

$$u = (\cos\theta \sin\psi\, ,\, \sin\theta \sin\psi\, ,\, \cos\psi)\, , \tag{6.5}$$

with $d\sigma(u) = \sin\psi \, d\psi d\theta$. We bound (6.4) by

$$C \sum_{j=1}^{N_0 2^{2k}} \int_{\mathbb{S}^1} \int_{D_{j,k}} \int_0^{+\infty} \int\int_{D^*_{j,k}} h(s)^{2\beta} \left| \hat{q}(-\frac{h(s)}{2}\; (u(\psi,\theta) + v)) \right|^2 \sin\psi \, d\psi d\theta \, s\, ds\, d\sigma(v) d\sigma(z)\, ,$$

where $u(\psi,\theta)$ is given by (6.5) and $D^*_{j,k} := \{(\psi,\theta) \in [0,\pi] \times [0, 2\pi) \,:\, u(\psi,\theta) \in \widetilde{D}_{j,k}\}$. We remark that for $(\psi,\theta) \in D^*_{j,k}$ it holds

$$|u(\psi,\theta) + v| \geq \frac{1}{100}. \tag{6.6}$$

For $z \in \mathbb{S}^1$, $j \in \{1, ..., N_0 2^{2k}\}$ and $v \in D_{j,k}$ fixed we consider the change $(s, \theta, \psi) \to (\lambda_1, \lambda_2, \lambda_3) = \lambda$ given by

$$\begin{aligned} \lambda = \xi + sz - \phi' - \xi' &= -\frac{h(s)}{2} \, (u(\psi,\theta) + v) \\ &= -\frac{h(s)}{2} \; (\, \cos\theta \sin\psi + v_1 \, , \, \sin\theta \sin\psi + v_2 \, , \, \cos\psi + v_3 \,) \; , \end{aligned}$$

where $v = (v_1, v_2, v_3)$. We have $d\lambda = \frac{sh(s) \sin\psi |1 + u(\psi,\theta)\cdot v|}{8} \, ds d\psi d\theta$. From condition (6.6) we deduce that $|\lambda| \sim h(s)$. We define this family of sets with overlapping constant independent of k contained in the interior of convex cones in $\mathbb{R}^3$:

$$H_{j,k} := \{ r(u+v) \,:\, u \in \widetilde{D}_{j,k} \, , \, v \in D_{j,k} \, , \, r < 0 \} \, .$$

Since $\sigma(D_{j,k}) \sim 2^{-2k}$, we have

$$\begin{aligned} F_k(\xi) &\leq C \sum_{j=1}^{N_0 2^{2k}} \int_{\mathbb{S}^1} \int_{D_{j,k}} \int_{H_{j,k}} |\lambda|^{2\beta - 1} |\hat{q}(\lambda)|^2 d\lambda \, d\sigma(v) d\sigma(z) \\ &\leq C \, 2^{-2k} \int_{\mathbb{R}^3} \left(\sum_{j=1}^{N_0 2^{2k}} \chi_{H_{j,k}}(\lambda) \right) |\lambda|^{2\beta-1} |\hat{q}(\lambda)|^2 d\lambda \\ &\leq C \, 2^{-2k} \int_{\mathbb{R}^3} |\lambda|^{2\beta - 1} |\hat{q}(\lambda)|^2 d\lambda \, . \end{aligned}$$

□

• Proof of (ii). We also express $\eta \in \Lambda^*(\xi)$ as $\eta = \xi + sz$, with $s > 0$, $z \in \{\xi\}^\perp$, $|z| = 1$. We use the same notation $h(s)$. Since $|\xi| \leq \left(\frac{\sqrt{2}}{2} + \frac{1}{100} \right) |\eta|$, it is true that $s \geq 0.9|\xi|$. It holds $d\sigma(\eta) = s ds d\sigma(z)$. For $z \in \mathbb{S}^1$ fixed, we take a reference $\{e_1, e_2, e_3\}$ in $\mathbb{R}^3$ such that $z = e_3$. Then $\xi = (\xi_1, \xi_2, 0)$ and $\eta = (\xi_1, \xi_2, s)$. We write for $\tau \in \mathcal{B}_\xi(\xi + sz)$,

$$\tau = \frac{\xi + sz}{2} + \frac{h(s)}{2} \, v \, , \quad v \in \mathbb{S}^2 \, ,$$

with $d\sigma(\tau) = C\, h(s)^2 d\sigma(v)$. We express v in spherical coordinates with respect of our reference:

$$v = (\sin\Theta\, \cos\theta, \sin\Theta\, \sin\theta, \cos\Theta), \quad 0 \le \Theta \le \pi\,,\ -\pi \le \theta < \pi\,,$$

where $d\sigma(v) = \sin\Theta\, d\Theta\, d\theta$. For s, z fixed, let

$$\mathcal{M}(s,z) := \{(\Theta,\theta) \in [0,\pi] \times [-\pi,\pi) \,:\, \tau(s,z,\Theta,\theta) \in \mathcal{B}_\xi(\xi + sz)\}\,.$$

We obtain

$$G(\xi) \le \int_{\mathbb{S}^1} \int_{0.9|\xi|}^{\infty} \int\int_{\mathcal{M}(s,z)} s\, h(s)^{2\beta+2\varepsilon} \sin\Theta$$
$$\times \left| \hat{q}\left(\frac{1}{2}\,(\xi_1 - h(s)\,\cos\theta\,\sin\Theta, \xi_2 - h(s)\,\sin\theta\,\sin\Theta, s - h(s)\,\cos\Theta)\right)\right|^2 d\Theta d\theta ds d\sigma(z)\,.$$

For each u, we take the change of variables $(s,\Theta,\theta) \to \lambda = (\lambda_1,\lambda_2,\lambda_3)$ given by

$$\lambda = \eta - \tau = \frac{1}{2}\,(\xi_1 - h(s)\,\cos\theta\,\sin\Theta, \xi_2 - h(s)\,\sin\theta\,\sin\Theta, s - h(s)\,\cos\Theta)\,.$$

It holds $d\lambda = \frac{h(s)}{8}\,\sin\Theta\,|s - h(s)\cos\Theta|\,ds\,d\Theta\,d\theta$.

Since $\tau \in \mathcal{B}_\xi(\xi + sz)$ we know that $|\tau| \le \left(\frac{1}{\sqrt{2}} + \frac{1}{50}\right)|\eta|$ and hence,

$$\left(1 - \left(\frac{1}{\sqrt{2}} + \frac{1}{50}\right)\right)|\eta| \le |\eta - \tau| \le |\eta|.$$

Since $(\Theta,\theta) \in \mathcal{M}(s,z)$ it holds $|\lambda| \sim h(s)$ ($|\eta - \tau| \sim |\eta|$). Moreover, $|\xi - \tau| \le \frac{|\eta|}{50}$ and the angle γ between $\eta - \xi$ and $\eta - \tau$ satisfies $|\cos\gamma| \ge C > 0$. That is, $|(\eta - \xi)\cdot(\eta - \tau)| \sim |\eta - \xi|\,|\eta - \tau|$. This says that $|s - h(s)\cos\Theta| \sim |\lambda|$, since $z\cdot(\eta - \tau) = \frac{s - h(s)\cos\Theta}{2}$. On our domain of integration we have $|s - h(s)\cos\Theta| \sim s$:

$$|s - h(s)\cos\Theta| \sim |\lambda| \sim h(s) \sim s,$$

where the condition $h(s) \sim s$ follows from $s \ge 0,9|\xi|$.

We conclude

$$G(\xi) \le C \int_{\mathbb{S}^1}\int_{\mathbb{R}^3} |\hat{q}(\lambda)|^2 |\lambda|^{2\beta - 1 + 2\varepsilon} d\lambda\, d\sigma(z)\,.$$

□

References

[1] S. Agmon, *Spectral properties of Schrödinger operators and scattering theory,* Ann. Sc. Norm. Super. Pisa (4), Vol. II, n. 2 (1975), pp. 151-218.

[2] J.A. Barceló, D. Faraco, A. Ruiz, A. Vargas, *Reconstruction of singularities from full scattering data by new estimates of bilinear Fourier multipliers,* Preprint.

[3] G. Beylkin, *Imaging of discontinuities in the inverse scattering problem by inversion of a casual generalized Radon transform,* J. Math. Phys., 26 (1985), pp. 99-108.

[4] I. Beltita, A. Mellin *Analysis of the quadratic term in the backscattering transformation* arXiv:0801.2230v1 [math.AP]

[5] I. Beltita, A. Mellin *Local smoothing for the backscattering transform* arXiv:0712.3865v1 [math.AP]

[6] D. Colton, R. Kress, *Integral Equation Methods in Scattering Theory,* John Wiley & Sons, New York, 1983.

[7] G. Eskin, J. Ralston, *The inverse backscattering problem in 3 dimension,* Comm. Math. Phys., 124 (1989), pp. 169-215.

[8] G. Eskin, J. Ralston, *Inverse backscattering in two dimensions,* Comm. Math. Phys., 138 (1991), pp. 451-486.

[9] G. Eskin, J. Ralston, *Inverse backscattering,* J. Anal. Math., 58 (1992), pp. 177-190.

[10] P. Grisvard, *Elliptic problems in nonsmooth domains,* Pitman Boston, 1985.

[11] A. Greenleaf, G. Uhlmann, *Recovery of singularities of a potential from singularities of the scattering data,* Comm. Math. Phys., 157 (1993), pp. 549-572.

[12] P. Hajlasz, *Sobolev spaces on an arbitrary metric space,* Potential Anal., 5, n. 4 (1996), pp. 403-415.

[13] C.E. Kenig, A. Ruiz, C.D. Sogge, *Uniform Sobolev inequalities and unique continuation for second order constant coefficients differential operators,* Duke Math. J., 55 (1987), pp. 329-347.

[14] R. Lagergren, *Backscattering in three dimensions,* Ph. D.Thesis, Lund University, 2001.

[15] A. Melin, *Some transforms in potential scattering in odd dimension,* Inverse problems and spectral theory, pp. 103-134, Contemp. Math., 348, Amer. Math. Soc., Providence, RI, 2004.

[16] A. Nachman, *Inverse scattering at fixed energy,* Proceedings of the 10th International Congress on Mathematical Physics, Leipzig, Springer Verlag, pp. 434-441, 1992.

[17] R.G. Novikov, *Multidimensional inverse spectral problem for the equation $-\Delta\Psi + (v(x) - Eu(x))\Psi = 0$,* Funct. Anal. Appl., 22, n. 4 (1988), pp. 263-272.

[18] P. Ola, L. Päivärinta, V. Serov, *Recovering singularities from backscattering in two dimensions,* Comm. Partial Differential Equations, 26, n. 3-4 (2001), pp. 697-715.

[19] L. Päivärinta, V. Serov, *Recovery of singularities of a multidimensional scattering potential,* SIAM J. Math. Anal., 29 (3) (1998), pp. 697-711.

[20] L. Päivärinta, V. Serov, E. Somersalo, *Reconstruction of singularities of a scattering potential in two dimensions,* Adv. in Appl. Math., 15 (1) (1994), pp. 97-113.

[21] L. Päivärinta, E. Somersalo, *Inversion of discontinuities for the Schrödinger equation in three dimensions,* SIAM J. Math. Anal., 22 (2) (1991), pp. 480-499.

[22] R.T. Prosser, *Formal solutions of inverse scattering problems,* J. Math. Phys., 23 (1982), pp. 2127-2130.

[23] A.G. Ramm, *Recovery of a potential from fixed-energy scattering data,* Inverse Problems, 4 (1988), pp. 877-886.

[24] J.M. Reyes, *Inverse backscattering for the Schrödinger equation in 2D,* Inverse Problems, 23 (2007), pp.625-643.

[25] J.M. Reyes, *Problema inverso de scattering para la ecuaciön de Schrödinger: Reconstrucción parcial del potencial a partir de datos de retrodispersión en 2D y 3D,* Ph.D. Thesis, Universidad Autónoma de Madrid, 2007: http://www.uam.es/gruposinv/inversos/publicaciones/index.html

[26] A. Ruiz, *Recovery of the singularities of a potential from fixed angle scattering data,* Comm. Partial Differential Equations, 26, n. 9-10 (2001), pp. 1721-1738.

[27] A. Ruiz, *Harmonic Analysis and Inverse Problems,* Notes of the 4th Summer School in Inverse Problems, Oulu, Finland, 2002: http://www.uam.es/gruposinv/inversos/publicaciones/index.html

[28] A. Ruiz, A. Vargas, *Partial recovery of a potential from backscattering data,* Comm. Partial Differential Equations, 30, n. 1-3 (2005), pp. 67-96.

[29] A. Ruiz, L. Vega, *On local regularity of Schrödinger equations,* Int. Math. Res. Not., 1 (1993), pp. 13-27.

[30] P. Stefanov, *Generic uniqueness for two inverse problems in potential scattering,* Comm. Partial Differential Equations, 17 (1992), pp. 55-68.

[31] Z. Sun, G. Uhlmann, *Recovery of singularities for formally determined inverse problems,* Comm. Math. Phys., 153 (1993), pp. 431-445.

[32] Z. Sun, G. Uhlmann, *Generic uniqueness for an inverse boundary value problem,* Duke Math. J., 62, n. 1 (1991), pp. 131-155.

[33] M.E. Taylor, *Pseudodifferential operators and nonlinear PDE,* Progress in Mathematics, Birkhauser, 1946.

[34] G. Uhlmann, *A time-dependent approach to the inverse backscattering problem,* Special issue to celebrate Pierre Sabatier's 65th birthday (Montpellier, 2000), Inverse Problems Vol. 17, n. 4 (2001), pp. 703-716.

[35] G.N. Watson, *The Theory of Bessel Functions,* Cambridge University Press, New York, 1948.